THE ELEMENTS OF ANALYTICAL GEOMETRY

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PART I. THE STRAIGHT LINE

By J. T. BROWN, M.A., B.Sc., and C. W. M. MANSON, M.A.

PART II. THE CIRCLE

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PARTS I. and II., in a single volume

PART III. CONIC SECTIONS

By J. T. BROWN, M.A., B.Sc.

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THE ELEMENTS OF
ANALYTICAL GEOMETRY

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PREFACE

This book is intended as an introduction to Analytical Geometry. The beginner will find notes and worked examples to guide him, and many simple exercises to give him practice in application. The needs of the more able reader are met by the inclusion of numerous carefully graded exercises throughout the text and by the provision of harder revision exercises.

Proofs of many of the earlier theorems are based on the expressions for the projections of a line on the axes of coordinates, and the equations of a straight line are derived largely by application of the gradient formula, as it has been found that the unity of treatment thus obtained is appreciated by the beginner. The more usual proofs given in the notes may be substituted but are included rather as worked examples and for the assistance which in some instances they afford the reader in memorising the results obtained.

The Straight Line, Circle and Conic Sections are dealt with in Parts I, II and III respectively, and the book covers all the Analytical Geometry generally required by the Senior Forms of Schools or the Junior Classes of Colleges or Universities.

J. T. B.

C. W. M. M.

NOTE TO PART II

In the text the gradients of tangents and normals are determined from first principles, but for the reader familiar with the calculus proofs by differentiation are given in the notes.

NOTE TO PART III

Part III is intended as an introduction to Analytical Conics. The introductory chapter establishes the standard equations, and contains simple exercises to assist the reader to gain some familiarity with the curves, their foci and directrices. The parabola, ellipse and hyperbola are treated in the subsequent chapters, and in each case parametric representation is introduced at the beginning as it is the writer's experience that learners benefit by having from the start the consequent choice of methods in attacking problems.

One chapter is devoted to polar coordinates and one to the general equation; the latter is discussed so far as is necessary for the tracing of conics.

The ground covered is that of the examinations for English Higher School Certificates; the first four chapters are sufficient for the Additional Geometry papers of the Scottish Leaving Certificate.

J. T. B.

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PART I.—THE STRAIGHT LINE

CHAPTER I

CARTESIAN AND POLAR COORDINATES

§ **1.** *Cartesian Coordinates.*

Let $X'OX$, $Y'OY$ (Fig. 1) be perpendicular lines and P any point in their plane. Let PM, PN be drawn perpendicular to $X'X$, $Y'Y$ respectively. Let $OM = x$ (positive or negative according as OM is in the direction of $X'X$ or XX') and let $ON = y$ (positive or negative according as ON is in the direction of $Y'Y$ or YY'). Then the position of P relative to $X'X$ and $Y'Y$ is known when x and y are known. x is called the abscissa, y the ordinate and x and y the Cartesian coordinates of the point P which is briefly referred to as the point (x, y).

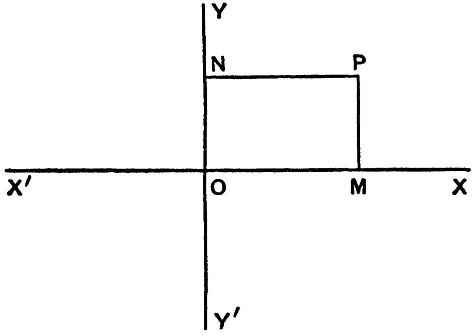

FIG. 1.

$X'X$ is called the x-axis, $Y'Y$ the y-axis, $X'X$ and $Y'Y$ the coordinate axes and O the origin.

Note 1. The coordinates are called Cartesian after Descartes (1596-1650), French philosopher and mathematician, who first

used this method of determining the position of any point in a plane.

Note 2. Since $MP = ON$ (Fig. 1), MP is often referred to as the ordinate of P.

Note 3. Rectangular axes, i.e. axes which are mutually perpendicular, are most commonly used; it is sometimes convenient, however, to use oblique axes, i.e. axes which are not mutually perpendicular, and then PM, PN are drawn parallel to the y- and x-axes respectively.

Note 4. The coordinate axes divide their plane into four sections called quadrants, which are numbered as in Fig. 2.

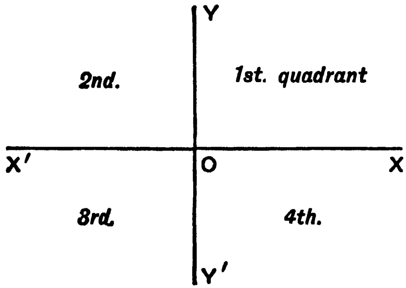

FIG. 2.

The reader should familiarise himself with the signs (Fig. 3) of coordinates of points in the four quadrants.

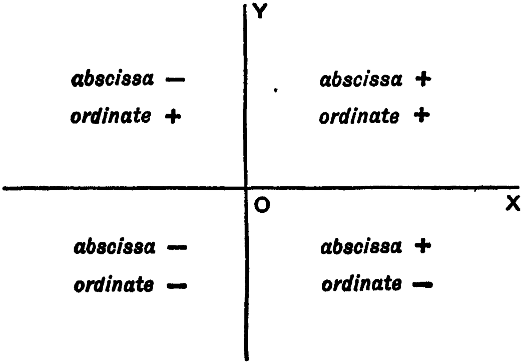

FIG. 3.

§ **2.** *If A, B are points on the x-axis, then with the convention of signs stated in* § 1,

$$AB = OB - OA,$$

where AB is positive or negative according as the direction of AB is that of $X'X$ or XX'.

There are six possible cases, as shown in Fig. 4.

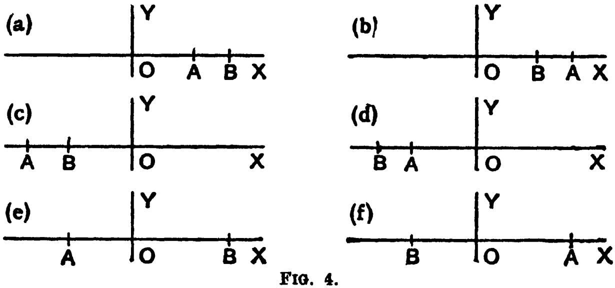

FIG. 4.

In Fig. 4 (*a*),
$$OB = OA + AB;$$
$$\therefore\ AB = OB - OA.$$

In Fig. 4 (*b*),
$$OA = OB + BA;$$
$$\therefore\ -BA = OB - OA;$$
$$\therefore\ AB = OB - OA.$$

The other cases are proved in a similar manner and are left as an exercise to the reader.

Note 1. " The straight line AB " may mean (i) a straight line of infinite length passing through A and B, (ii) a straight line of finite length terminated by A and B, (iii) the finite line AB with sense or direction taken into account as well as magnitude. In case (iii) we speak of " the step AB."

The relation $AB = OB - OA$ holds for steps along any line.

Note 2. If x_1, x_2 are the abscissae of A, B (Fig. 4),

$$AB = x_2 - x_1.$$

Similarly if y_1, y_2 are the ordinates of points A, B on the y-axis,

$$AB = y_2 - y_1.$$

Note 3. The expressing of a step AB in terms of the distances of A and B from a third point O in the line AB is called inserting the origin O.

Example. The abscissae of A, B, two points on the x-axis, are x_1, x_2 respectively; the point P divides AB in the ratio $k:l$; prove that the abscissa of P is

$$\frac{kx_2+lx_1}{k+l}$$

and state the corresponding relation when A, B lie on the y-axis.

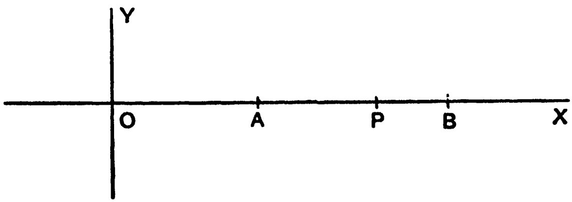

FIG. 5.

Let the abscissa of P (Fig. 5) be x.

Then
$$\frac{k}{l}=\frac{AP}{PB}$$
$$=\frac{OP-OA}{OB-OP}$$
$$=\frac{x-x_1}{x_2-x};$$
$$\therefore\ kx_2+lx_1=(k+l)x;$$
$$\therefore\ x=\frac{kx_2+lx_1}{k+l}.$$

If the points lie on the y-axis, we have the corresponding relation,

$$y=\frac{ky_2+ly_1}{k+l},$$

where y is the ordinate of P.

§ 3. *Polar Coordinates.*

Let OA (Fig. 6) be a given line and P any point in a given plane. Let θ be the angle through which a line rotates in

moving from the position OA into the position OP, θ being positive or negative according as the direction of rotation is counter-clockwise or clockwise. Then, if r is the length of OP, the position of P is known when r and θ are known.

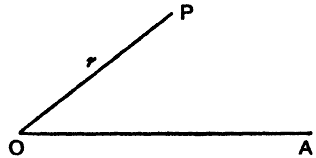

FIG. 6.

r is called the radius vector, θ the vectorial angle and r, θ the polar coordinates of the point P, which is briefly referred to as the point (r, θ). OA is called the initial line and O the pole.

In the foregoing definition of polar coordinates it has been assumed that r is positive; it is sometimes useful to affix a sign to r taking it as positive when measured along the arm of the angle θ and negative when it is measured in the opposite direction.

Note. The same point may be determined by different pairs of polar coordinates, e.g. (r, θ), $(-r, \theta+\pi)$, $(r, \theta+2\pi)$.

§ 4. *Relations between Cartesian and Polar Coordinates.*

Let P (Fig. 7) be the point (x, y) with reference to rectangular axes $X'OX$, $Y'OY$, and the point (r, θ) with reference to the pole O and initial line OX.

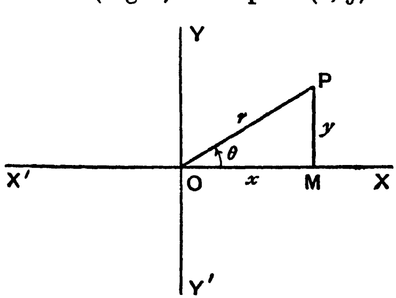

FIG. 7.

From the definition of the circular functions, we have

$$\frac{x}{r} = \cos\theta,$$

i.e. $$x = r\cos\theta,$$

and similarly

$$y = r\sin\theta.$$

Hence $$\tan\theta = \frac{y}{x}$$

and $$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2.$$

These relations enable us to change from one system of coordinates to the other.

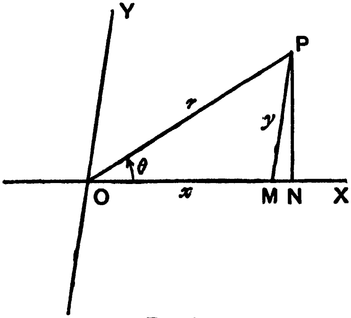

FIG. 8.

Note. In the case of axes inclined at angle ω, let PN (Fig. 8) perpendicular to OX meet OX at N; then $N\hat{M}P=\omega$ and

$$\begin{aligned} r\cos\theta &= ON \\ &= OM+MN \\ &= x+y\cos\omega, \end{aligned}$$

while

$$\begin{aligned} r\sin\theta &= NP \\ &= y\sin\omega; \end{aligned}$$

$$\therefore\ \tan\theta=\frac{y\sin\omega}{x+y\cos\omega}$$

and $$r^2=x^2+y^2+2xy\cos\omega.$$

Example 1. *Find the polar coordinates of the point whose Cartesian coordinates are* $(-\sqrt{3}, 1)$.

We have $$r\cos\theta=-\sqrt{3}$$

and $$r\sin\theta=1.$$

Taking r positive, θ lies in the second quadrant,

and $$\tan\theta=-\frac{1}{\sqrt{3}};$$

$$\therefore\ \theta=\frac{5\pi}{6}.$$

Also $$r^2=(-\sqrt{3})^2+(1)^2;$$

$$\therefore\ r=2,$$

i.e. the polar coordinates are $\left(2, \frac{5\pi}{6}\right)$.

Example 2. *Find the Cartesian coordinates of the point whose polar coordinates are* $\left(\sqrt{2}, \frac{5\pi}{4}\right)$.

We have $$\begin{aligned} x &= \sqrt{2}\cos\frac{5\pi}{4} \\ &= -1, \end{aligned}$$

and $$y = \sqrt{2} \sin \frac{5\pi}{4}$$
$$= -1,$$

i.e. the Cartesian coordinates are $(-1, -1)$.

§ **5.** *If a line AB of length r is inclined to the x-axis at angle θ, and if MA, NB are the ordinates of A, B,*

$$MN = r \cos \theta.$$

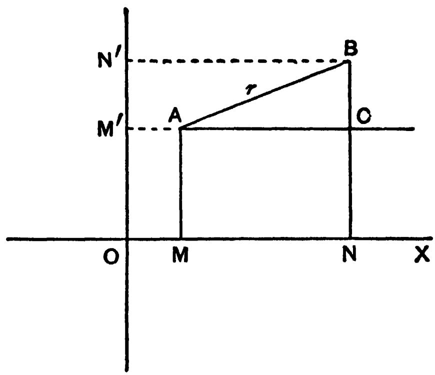

FIG. 9.

Let AC (Fig. 9) be parallel to OX and cut NB at C.

Then $$C\hat{A}B = \theta,$$
but $$MN = AC;$$
$$\therefore \qquad = r \cos \theta.$$

Note 1. If OM', ON' (Fig. 9) are the ordinates of A, B,

$$M'N' = r \sin \theta.$$

Note 2. If A, B are the points (x_1, y_1), (x_2, y_2), we have

$$r \cos \theta = MN \;\; = x_2 - x_1,$$
$$r \sin \theta = M'N' = y_2 - y_1.$$

These are important relations and should be remembered.

Note 3. It should be noted that MN, $M'N'$ are the projections of AB on the x- and y-axes respectively.

Note 4. If the axes are inclined at angle ω,

$$r \cos \theta = (x_2 - x_1) + (y_2 - y_1) \cos \omega,$$
$$r \sin \theta = (y_2 - y_1) \sin \omega,$$

and $M'N'$ is then the projection of AB on a line perpendicular to the x-axis.

Example. *A, B (Fig. 9) are the points $(-4, -1)$, $(6, -5)$; determine the magnitude and sign of the projections MN, $M'N'$.*

$$MN = 6 - (-4) = 10$$

and $$M'N' = -5 - (-1) = -4.$$

Exercises

1. Plot the points $(3, 2)$, $(-2, 2)$, $(-2, -1)$, $(3, -1)$ and show that they are the vertices of a rectangle.

2. Plot the points $(4, 3)$, $(-1, 3)$, $(-1, -2)$, $(4, -2)$, and show that they are the vertices of a square.

3. Show that the points $(6, 1)$, $(6, 4)$, $(-1, 4)$ are vertices of a rectangle and find the fourth vertex.

4. Show that the points $(1, 2)$, $(-3, 2)$, $(-3, -2)$ are vertices of a square and find the coordinates of the fourth vertex.

5. Show that, if N is the foot of the ordinate of the point $P(x, y)$ and PN is produced its own length to Q, the coordinates of Q are x and $-y$.

6. Show that, if O is the origin, P the point (x, y) and PO is produced its own length to Q, Q is the point $(-x, -y)$.

7. Show that the point $\left(\frac{x}{2}, \frac{y}{2}\right)$ is the mid-point of the line joining the origin to the point (x, y).

8. Find the magnitude and sign of the step AB when A, B are the points:

(i) $(3, 0)$, $(1, 0)$. (iii) $(0, -4)$, $(0, -1)$. (v) $(-3, 0)$, $(4, 0)$.
(ii) $(2, 0)$, $(-3, 0)$. (iv) $(0, -2)$, $(0, -6)$. (vi) $(-5, 0)$, $(-1, 0)$.

9. A, B are the points $(x_1, 0)$, $(x_2, 0)$, M the mid-point of AB; prove that the abscissa of M is $\frac{x_1 + x_2}{2}$.

10. Find the ordinates of the points of trisection of the line joining the points $(0, -3)$, $(0, 4\frac{1}{2})$.

11. A, P are the points $(-2, 0)$, $(5, 0)$; P divides AB in the ratio $7:3$; find the abscissa of B.

12. A, B are the points $(x_1, 0)$, $(x_2, 0)$ and P divides AB externally in the ratio $k:l$; show that the abscissa of P is $\dfrac{kx_2 - lx_1}{k-l}$.

13. A, B are the points $(0, -4)$, $(0, 1)$; P, Q divide AB internally and externally in the ratio $3:2$; find the coordinates of P and Q and the length of PQ.

14. Find the Cartesian coordinates of the points

$$\left(2, \frac{\pi}{3}\right), \left(2\sqrt{2}, \frac{3\pi}{4}\right), \left(3, \frac{3\pi}{2}\right), \left(1, \frac{11\pi}{6}\right).$$

15. Find the polar coordinates of the points $(\sqrt{3}, 1)$, $(1, -1)$, $(-1, -\sqrt{3})$, $(0, 1)$.

16. O is the origin, P the point $(-2, 2\sqrt{3})$; show that the length of OP is 4.

17. Find the distances of the following points from the origin,

$$(3, -4), (-5, -12), (-8, 6), \left\{\frac{a(1-t^2)}{1+t^2}, \frac{2at}{1+t^2}\right\}.$$

18. Find the inclination to the x-axis of the line joining the origin to the point $(-\sqrt{3}, 1)$.

19. P is the point $(1, 2)$ relative to axes OX, OY where $X\hat{O}Y$ is acute and $\tan X\hat{O}Y = \frac{5}{12}$; show that $\tan X\hat{O}P = \frac{10}{37}$.

20. Find the distance of the point $(2, 5)$ from the origin, the axes being inclined at the acute angle with sine $\frac{3}{5}$.

21. Prove that the distance between the points whose polar coordinates are (r_1, θ_1), (r_2, θ_2) is

$$\sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}.$$

22. Find the distance between the points whose polar coordinates are $\left(2, \frac{\pi}{6}\right)$, $\left(1, \frac{4\pi}{3}\right)$.

23. Find the lengths of the projections on the coordinate axes of the line PQ where P, Q are respectively the points:

(i) $(3, 2)$, $(6, 5)$.
(ii) $(2, 1)$, $(7. 1)$.
(iii) $(-2, 3)$, $(-8, -2)$.
(iv) $(-2, -1)$, $(5, -4)$.

24. P, Q are the points $(-1, 3)$, $(5, -2)$ relative to oblique axes inclined at angle ω where $\cos\omega = \frac{3}{5}$; find the length of the projection of PQ on the x-axis.

25. A, B, C, D are the points $(-2, 4)$, $(3, 2)$, $(-3, -1)$, $(2, 3)$: show that the projections of AB and CD on the x-axis are equal.

26. A, B, C, D are the points $(4, 1)$, $(-1, -2)$, $(0, 2)$, $(-5, -1)$; show that the projections of AB and CD (i) on the x-axis, (ii) on the y-axis are equal.

CHAPTER II

DISTANCE FORMULA. SECTION FORMULAE. GRADIENT

§ 6. *Distance Formula.*

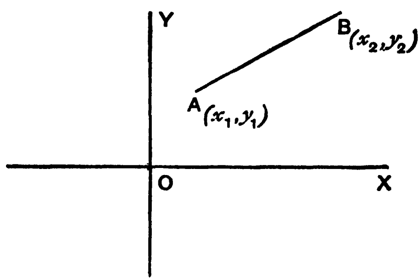

FIG. 10.

Let A, B (Fig. 10) be the points (x_1, y_1), (x_2, y_2), r the length of AB and θ the inclination of AB to OX.

Then
$$r\cos\theta = x_2 - x_1,$$
$$r\sin\theta = y_2 - y_1;$$
$$\therefore\ (x_2 - x_1)^2 + (y_2 - y_1)^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2;$$
$$\therefore\ r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note 1. r is considered positive; the formula gives the numerical distance between A and B.

Note 2. The expression for r may be written

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Note 3. In any particular case the expression for r may be obtained in the following manner:

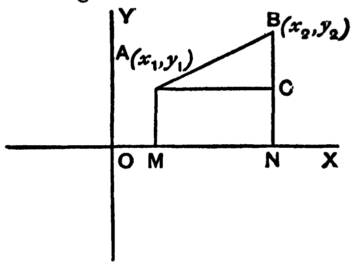

FIG. 11.

Let MA, NB (Fig. 11) be the ordinates of $A(x_1, y_1)$, $B(x_2, y_2)$, and let AC, parallel to OX, meet NB at C.

Then
$$\begin{aligned} r^2 &= AB^2 \\ &= AC^2 + CB^2 \\ &= (ON - OM)^2 + (NB - NC)^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2, \quad \text{for } NC = MA. \end{aligned}$$

Note 4. In the case of axes inclined at angle ω,
$$r \cos\theta = (x_2 - x_1) + (y_2 - y_1)\cos\omega,$$
$$r \sin\theta = (y_2 - y_1)\sin\omega$$
and
$$\therefore\ r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega}.$$

Example 1. *Find the distance between the points* $(-2, 1)$, $(4, 9)$.

$$\begin{aligned} \text{Distance} &= \sqrt{(4+2)^2 + (9-1)^2} \\ &= \sqrt{36 + 64} \\ &= 10. \end{aligned}$$

Example 2. *Show that the points*

$$(1, -1),\ (-1, 1),\ (-\sqrt{3}, -\sqrt{3})$$

are the vertices of an equilateral triangle.

Letting the points in order be A, B, C, we have

$$AB^2 = 2^2 + 2^2 = 8,$$
$$BC^2 = (\sqrt{3} - 1)^2 + (\sqrt{3} + 1)^2 = 8$$
and
$$CA^2 = (\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 = 8;$$
$$\therefore\ \triangle ABC \text{ is equilateral.}$$

§ 7. *Section Formulae.*

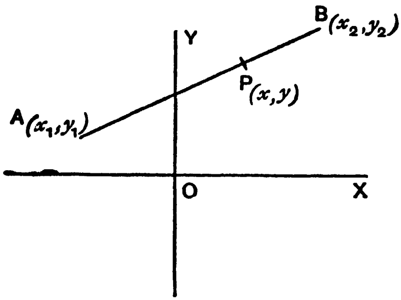

FIG. 12.

Let A, B (Fig. 12) be the points (x_1, y_1), (x_2, y_2), and θ the inclination of AB to the x-axis. Let P (x, y) be the point on AB such that $AP : PB = k : l$.

Then
$$AP \cos\theta = x - x_1,$$
$$PB \cos\theta = x_2 - x,$$

and $\therefore$
$$\frac{x - x_1}{x_2 - x} = \frac{AP}{PB}$$
$$= \frac{k}{l};$$
$$\therefore \ (k + l)x = kx_2 + lx_1;$$
$$\therefore \ x = \frac{kx_2 + lx_1}{k + l},$$

and similarly
$$y = \frac{ky_2 + ly_1}{k + l}.$$

Note 1. The reader should note that, in the section formulae, k, which corresponds to the segment AP, is multiplied by the coordinates of B, while l, which corresponds to PB, is multiplied by the coordinates of A.

Note 2. Putting $k = l$, we find that the mid-point of the line joining the points (x_1, y_1), (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

Example. Find the coordinates of P which divides in the ratio 3 : 2 the line joining $A(-7, 1)$ and $B(3, 6)$.

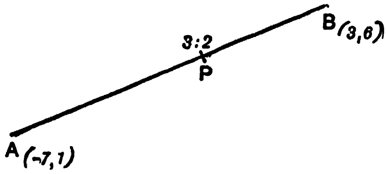

FIG. 13.

At P (Fig. 13),
$$x=\frac{3(3)+2(-7)}{3+2}=-1,$$
$$y=\frac{3(6)+2(1)}{3+2}=4.$$

§ **8.** *Section Formulae (continued).*

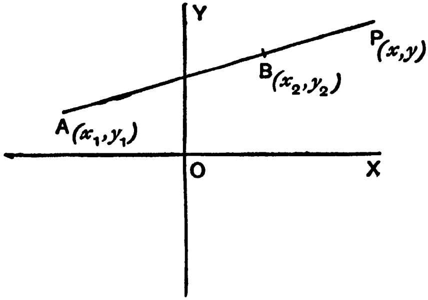

FIG. 14.

Let A, B (Fig. 14) be the points (x_1, y_1), (x_2, y_2) and θ the inclination of AB to the x-axis. Let $P(x, y)$ be the point on AB such that $AP : BP = k : l$.

Then
$$AP \cos\theta = x - x_1,$$
$$BP \cos\theta = x - x_2,$$
and $\therefore$
$$\frac{x-x_1}{x-x_2}=\frac{AP}{BP}$$
$$=\frac{k}{l};$$

$$\therefore \ (k-l)x = kx_2 - lx_1;$$

$$\therefore \ x = \frac{kx_2 - lx_1}{k-l},$$

and similarly

$$y = \frac{ky_2 - ly_1}{k-l}.$$

Note 1. As in the formulae for internal section, k which corresponds to the segment AP is multiplied by the coordinates of B, and l which corresponds to BP is multiplied by the coordinates of A.

Note 2. It has been assumed in §§ 7 and 8 that k and l are positive numbers; if, however, we take $AP : PB$ to be positive when P is an internal point and negative when it is external, we have in both cases the formulae

$$x = \frac{kx_2 + lx_1}{k+l}, \quad y = \frac{ky_2 + ly_1}{k+l}.$$

Example 1. *Find the coordinates of the point which divides externally in the ratio* 4 : 3 *the line joining the points* $(-3, -7)$, $(-1, -4)$.

$$\text{Abscissa} = \frac{4(-1) - 3(-3)}{4-3} = 5,$$

$$\text{ordinate} = \frac{4(-4) - 3(-7)}{4-3} = 5.$$

Example 2. *Find the coordinates of the point which divides externally in the ratio* 2 : 3 *the line joining the points* (0, 2), (4, 8).

$$\text{Abscissa} = \frac{2(4) - 3(0)}{2-3} = -8,$$

$$\text{ordinate} = \frac{2(8) - 3(2)}{2-3} = -10.$$

§ 9. *Gradient.*

The gradient of a line is defined to be the tangent of the angle which the line makes with the positive direction of the x-axis.

It follows that (i) if two lines are parallel, their gradients are equal, for the lines must be equally inclined to the positive direction of the x-axis; (ii) if two lines are perpendicular, the

product of their gradients is -1, for if one line is inclined at angle θ to the x-axis the other must be inclined at $\theta+\frac{\pi}{2}$, hence their gradients are

$$\tan\theta \quad \text{and} \quad \tan\left(\theta+\frac{\pi}{2}\right),$$

i.e.
$$\tan\theta \quad \text{and} \quad -\cot\theta\,;$$

$\therefore$ the product of the gradients is -1.

The gradient of the line joining any two points (x_1, y_1), (x_2, y_2) is

$$\frac{y_2-y_1}{x_2-x_1},$$

for, if the join of the points have length r and be inclined to the x-axis at angle θ,

$$r\cos\theta = x_2-x_1,$$
$$r\sin\theta = y_2-y_1$$

and $\therefore$
$$\tan\theta = \frac{y_2-y_1}{x_2-x_1}.$$

Note 1. The expression for the gradient may be written

$$\frac{y_1-y_2}{x_1-x_2}.$$

Note 2. In any particular case the expression for the gradient may be obtained in the following manner :

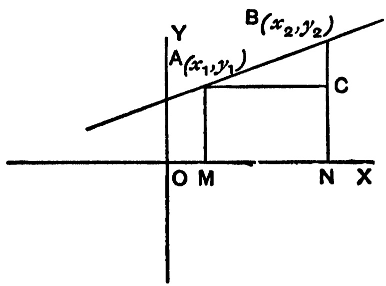

FIG. 15.

Let MA, NB (Fig. 15) be the ordinates of the points (x_1, y_1), (x_2, y_2) and let AC, parallel to OX, meet NB at C.

Then, if AB makes angle θ with OX, $C\hat{A}B=\theta$

and $\therefore$
$$\tan\theta=\frac{CB}{AC}$$
$$=\frac{NB-MA}{ON-OM},$$

i.e.
$$\text{gradient of } AB=\frac{y_2-y_1}{x_2-x_1}.$$

Note 3. In the case of axes inclined at angle ω,

$$r\cos\theta=(x_2-x_1)+(y_2-y_1)\cos\omega,$$
$$r\sin\theta=(y_2-y_1)\sin\omega$$

and $\therefore$
$$\tan\theta=\frac{(y_2-y_1)\sin\omega}{(x_2-x_1)+(y_2-y_1)\cos\omega}.$$

Note 4. The necessary and sufficient condition for three points A, B, C being collinear is that the gradient of AB = gradient of BC.

Example 1. *A, B, C, D are the points* $(-1, 3)$, $(1, 0)$, $(4, 2)$, $(0, 8)$; *prove that $AB \perp BC$, and that $AB \parallel DC$.*

$$\text{Gradient of } AB=\frac{3-0}{-1-1}=-\frac{3}{2},$$
$$\text{,, ,, } BC=\frac{0-2}{1-4}=\frac{2}{3},$$
$$\text{,, ,, } DC=\frac{2-8}{4-0}=-\frac{3}{2};$$
$$\therefore\ AB\perp BC \text{ and } \parallel DC.$$

Example 2. *A, B, C are the points* $(5, 7)$, $(-3, 1)$, $(-7, -2)$; *prove A, B, C collinear.*

$$\text{Gradient of } AB=\frac{1-7}{-3-5}=\frac{3}{4},$$
$$\text{,, ,, } BC=\frac{-2-1}{-7+3}=\frac{3}{4};$$
$$\therefore\ A, B, C \text{ are collinear.}$$

§ **10.** *Angle between two lines.*

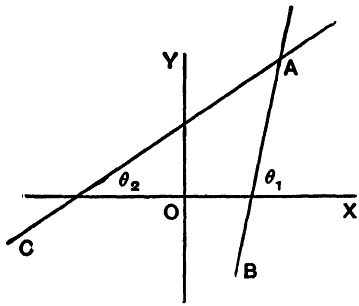

FIG. 16.

Let AB, AC (Fig. 16) have gradients m_1, m_2, and be inclined to the x-axis at angles θ_1, θ_2; then

$$C\hat{A}B = \theta_1 - \theta_2$$

$$\therefore \tan C\hat{A}B = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$= \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Note 1. The tangent of the supplement of $C\hat{A}B$ is $-\tan C\hat{A}B$; $\therefore$ the tangent of the angle between lines of gradients m_1, m_2 is

$$\pm\frac{m_1 - m_2}{1 + m_1 m_2}.$$

Note 2. The condition that two lines should be parallel or perpendicular may be deduced from the expression

$$\frac{m_1 - m_2}{1 + m_1 m_2},$$

for, if the lines are parallel, the tangent of the angle between them is zero and $\therefore m_1 = m_2$, and, if the lines are perpendicular, the tangent of the angle between them is infinite and $\therefore m_1 m_2 = -1$.

Example. Find the acute angle between two lines with gradients $-\frac{1}{2}$ *and* $\frac{1}{3}$.

$$\text{Tangent of angle} = \pm\frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{6}}$$

$$= \mp 1;$$

$\therefore$ the acute angle between the lines is 45°.

Exercises

1. Find the distance between the points:

(i) $(-1, 2), (3, -1)$. (iii) $(-7, 6), (2, 3)$.

(ii) $(3, -5), (8, 7)$. (iv) $(-3, -6), (-14, -8)$.

2. Prove that the points $(7, 3), (5, 5), (-1, -3)$ are the vertices of an isosceles triangle, and find the length of the base.

3. Prove that the points $(3, -2), (7, 6), (-1, 2), (-5, -6)$ are the vertices of a rhombus, and find the lengths of the diagonals.

4. Show that the point $(-1, -2)$ is the centre of a circle passing through the points $(11, 3), (-1, -15), (-13, -7), (4, -14)$.

5. Prove that the distance between the points $(at^2, 2at), \left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ is $a\left(t+\frac{1}{t}\right)^2$.

6. Prove that the distance between the points $(a \cos \theta, a \sin \theta)$ and $(a \cos \varphi, a \sin \varphi)$ is $2a \sin \frac{\theta - \varphi}{2}$.

7. Write down the coordinates of the mid-points of the lines joining the points:

(i) $(0, 0), (6, 8)$. (ii) $(-4, 3), (2, 5)$. (iii) $(0, 7), (-4, -3)$.

8. A, B, C, D are the points $(-2, 2), (3, 1), (5, 4), (-4, -1)$ respectively; show that AB and CD bisect each other.

9. P, Q are the points $(-1, 3), (2, -4)$ relative to axes inclined at 60°; show that $PQ = \sqrt{37}$.

10. Show that the points $(1, -1), (7, 3), (3, 5), (-3, 1)$ are the vertices of a parallelogram, and find the lengths of the diagonals.

11. Show that the origin is a point of trisection of the line joining the points $(-3, -2), (6, 4)$, and find the other point of trisection.

12. Find the coordinates of the points which divide internally and externally in the ratio $2 : 1$ the line joining the points $(-7, 7)$, $(-1, -2)$.

13. Find the coordinates of the points P, Q which divide internally and externally in the ratio $2 : 3$ the line joining the points $A(8, 10)$, $B(18, 20)$, and show that $MP \,.\, MQ = MB^2$ where M is the mid-point of AB.

14. A, B, C are the points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$; D is the midpoint of BC, and G divides AD in the ratio $2 : 1$; find the coordinates of G and show that the medians of $\triangle ABC$ are concurrent.

15. Find the gradients of the lines joining the points:

(i) $(1, 3), (5, 6)$. (iv) $\left(cp, \frac{c}{p}\right), \left(cq, \frac{c}{q}\right)$.

(ii) $(-1, 2), (3, 4)$. (v) $(ap^2, 2ap), (aq^2, 2aq)$.

(iii) $(-2, 1), (2, -3)$. (vi) $(a \cos \theta, b \sin \theta), (a \cos \varphi, b \sin \varphi)$.

16. A, B, C, D are the points $(-2, 0), (4, 3), (1, -1), (5, 1)$; prove that $AB \parallel CD$.

17. A, B, C, D are the points $(1, 2), (3, -2), (4, -1), (-2, -4)$; prove that $AB \perp CD$.

18. Show that the points $(1, 4), (-4, -1), (2, 1)$ are the vertices of a right-angled triangle.

19. Prove that the points $(4, 1), (1, 6), (-4, 3), (-1, -2)$ are the vertices of a square.

20. Prove that the points $(4, 0), (6, 1), (4, 3), (3, 2)$ are concyclic.

21. Prove that the points $(1, 1), (4, 5), (-2, -3)$ are collinear.

22. Show that the point $P(-2, 0)$ lies on the line passing through $A(7, -3)$ and $B(-5, 1)$; find the ratio $AP : PB$.

23. Show that the points $(a, 0)$, $(at^2, 2at)$, $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ are collinear for all values of t.

24. Find the angle between two lines with gradients $-3, 2$.

25. Find the tangent of the angle between two lines with gradients $-2, -\frac{1}{4}$.

26. A, B, C are the points $(-3, 2), (1, -5), (5, 3)$; find the gradients of AB and BC and the tangent of $A\hat{B}C$.

27. A, B, C are the points $(-1, -2), (2, 4), (-3, 2)$ relative to axes inclined at $60°$; find the tangent of $A\hat{B}C$.

28. A, B are the points $(-8, 0), (-2, 0)$; a point $P(x, y)$ is such that $PA = 2PB$; show that

$$x^2 + y^2 = 16.$$

29. The point (x, y) is any point on the line passing through the points $(1, 2), (3, 4)$; show that

$$x - y + 1 = 0.$$

30. The point (x, y) is any point on the line which passes through the point $(2, 3)$ and which is perpendicular to the line joining the points $(-1, 2), (-5, 4)$; show that

$$2x - y - 1 = 0.$$

CHAPTER III

AREAS

§ **11.** *The area of* $\triangle OAB$ *where* O *is the origin and* A, B *the points* (x_1, y_1), (x_2, y_2).

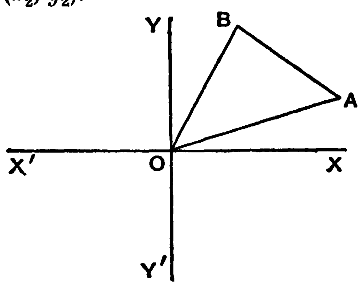

FIG. 17.

Let the polar coordinates of A, B (Fig. 17) be (r_1, θ_1), (r_2, θ_2).
Then area of

$$\begin{aligned}\triangle OAB &= \tfrac{1}{2}OA\,.\,OB\sin A\hat{O}B\\ &= \tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1)\\ &= \tfrac{1}{2}r_1r_2(\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1)\\ &= \tfrac{1}{2}(r_1\cos\theta_1\,.\,r_2\sin\theta_2-r_2\cos\theta_2\,.\,r_1\sin\theta_1)\\ &= \tfrac{1}{2}(x_1y_2-x_2y_1).\end{aligned}$$

Note 1. The reader should verify, by considering different positions of A and B, that, according as the order of the vertices $(0, 0)$, (x_1, y_1), (x_2, y_2) is counter-clockwise or clockwise, the expression $\frac{1}{2}r_1r_2\sin(\theta_2-\theta_1)$, and therefore the expression $\frac{1}{2}(x_1y_2-x_2y_1)$, is positive or negative.

Note 2. If we consider the area of a triangle to be positive when its vertices are named in counter-clockwise order, and negative when named in clockwise order, the expression $\frac{1}{2}(x_1y_2-x_2y_1)$ gives the area of $\triangle OAB$ in magnitude and sign.

Note 3. In the case of oblique axes, inclined at angle ω,

$$r_1 \cos\theta_1 = x_1 + y_1 \cos\omega,$$
$$r_1 \sin\theta_1 = y_1 \sin\omega$$

and $\therefore$ $$\triangle OAB = \tfrac{1}{2}(x_1y_2 - x_2y_1)\sin\omega.$$

Example. Find the area of the triangle whose vertices are the origin and the points (3, 4), (2, 6).

$$\text{Area} = \tfrac{1}{2}(3 \times 6 - 4 \times 2)$$
$$= 5.$$

§ **12.** *If O is any point in the plane of $\triangle ABC$, it follows from the convention of signs* (*see* § 11, *Note* 2) *that*

$$\triangle ABC = \triangle OAB + \triangle OBC + \triangle OCA.$$

The point O may lie

(*a*) within the triangle ;

(*b*) on a side of the triangle ;

(*c*) outside the triangle, and within the arms of an angle of the triangle ;

(*d*) outside the triangle, and within the arms of an angle vertically opposite to an angle of the triangle.

These four cases are shown in Fig. 18.

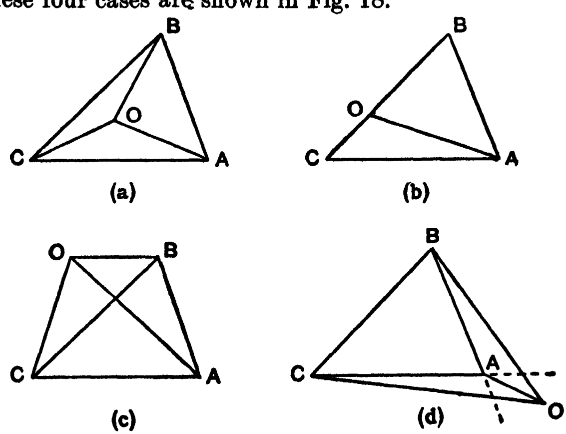

FIG. 18.

Joining O to A, B, C the theorem follows at once in cases (*a*), (*b*).

In case (*c*) $\quad \triangle ABC = \triangle OAB + \triangle OCA - \triangle OCB$

$$= \triangle OAB + \triangle OBC + \triangle OCA.$$

In case (*d*) $\quad \triangle ABC = \triangle OBC - \triangle OBA - \triangle OAC$

$$= \triangle OAB + \triangle OBC + \triangle OCA.$$

Note 1. The reader should prove that the theorem holds when O is a vertex of $\triangle ABC$.

Note 2. In Fig. 18, $\triangle ABC$ is positive; if $\triangle ABC$ is taken as negative, all the triangles named in the proof change sign, and hence the theorem still holds.

Note 3. Expressing $\triangle ABC$ in terms of triangles with a common vertex at O, is called inserting the origin O. (Cf. § 2, Note 3.)

§ 13. *The area of the triangle with vertices* $(x_1, y_1), (x_2, y_2), (x_3, y_3)$.

Let the given vertices be A, B, C respectively.

Inserting the origin O, we have in magnitude and sign

$$\begin{aligned}\triangle ABC &= \triangle OAB + \triangle OBC + \triangle OCA \\ &= \tfrac{1}{2}\{(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)\} \\ &= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3).\end{aligned}$$

Note 1. The expression may be written in the form

$$\tfrac{1}{2}\{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\}.$$

Note 2. In any particular case the expression for the area of $\triangle ABC$ may be obtained thus:

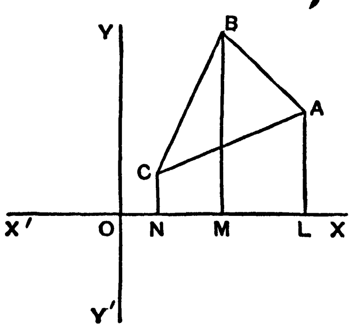

FIG. 19

Let L, M, N (Fig. 19) be the feet of the ordinates of the points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ respectively.

$$\text{Area } MLAB = \tfrac{1}{2}(LA + MB)ML$$
$$= \tfrac{1}{2}(y_1 + y_2)(x_1 - x_2).$$

Similarly $\qquad NMBC = \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3),$

and $\qquad NLAC = \tfrac{1}{2}(y_1 + y_3)(x_1 - x_3).$

But
$$\triangle ABC = MLAB + NMBC - NLAC$$
$$= \tfrac{1}{2}\{(y_1 + y_2)(x_1 - x_2) + (y_2 + y_3)(x_2 - x_3) - (y_1 + y_3)(x_1 - x_3)\}$$
$$= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3).$$

Note 3. By employing a method similar to that of Note 2, the reader should show that $\triangle OAB = \tfrac{1}{2}(x_1y_2 - x_2y_1)$.

Note 4. In the case of oblique axes, inclined at angle ω,
$$\triangle ABC = \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)\sin\omega.$$

Note 5. *The condition that the points* (x_1, y_1), (x_2, y_2), (x_3, y_3) *should be collinear.*

Let the given points be A, B, C respectively.

A, B, C are collinear if the area of $\triangle ABC$ is zero,

i.e. if $\qquad x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3 = 0.$

Example 1. *Find the area of the triangle with vertices* $(-3, -2)$, $(1, 4)$, $(2, 3)$.

$$\text{Area} = \tfrac{1}{2}(-12 + 3 - 4 + 2 - 8 + 9)$$
$$= -5.$$

The negative sign shows that the vertices were given in clockwise order.

Example 2. *Find the area of the triangle with vertices* $A(3, 1)$, $B(2t, 3t)$, $C(t, 2t)$, *and show that* A, B, C *are collinear when* $t = -2$.

$$\triangle ABC = \tfrac{1}{2}(9t + 4t^2 + t - 2t - 3t^2 - 6t)$$
$$= \tfrac{1}{2}(t^2 + 2t)$$

$$=\frac{t}{2}(t+2)$$

$$=0 \text{ if } t=0 \text{ or } t=-2.$$

When $t=0$, B and C coincide; when $t=-2$, A, B, C are collinear.

§ **14.** *The area of the quadrilateral, whose vertices taken in order are* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$.

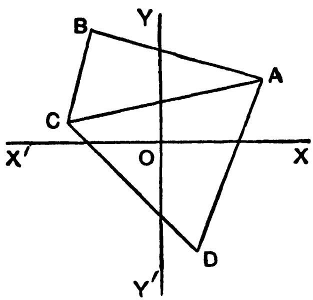

FIG. 20.

Let $ABCD$ (Fig. 20) be the given quadrilateral. Join AC.

Area $ABCD$

$$= \triangle ABC + \triangle ACD$$

$$= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$
$$+ \tfrac{1}{2}(x_1y_3 + x_3y_4 + x_4y_1 - x_3y_1 - x_4y_3 - x_1y_4)$$

$$= \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4).$$

Note 1. The expression for the area will be positive or negative according as A, B, C, D are in counter-clockwise or clockwise order.

Note 2. The expression for the area holds only if the vertices are taken in order; if therefore the order of the vertices is not known, it must be determined by plotting the given points.

Example. Find the area of the quadrilateral with vertices $A(2, 5)$, $B(4, 1)$, $C(-4, 2)$, $D(1, -1)$.

The vertices in order (Fig. 21) are $ACDB$;

$\therefore$ area of quadrilateral $= \frac{1}{2}(4+4+1+20+20-2+4-2)$

$= \frac{49}{2}$.

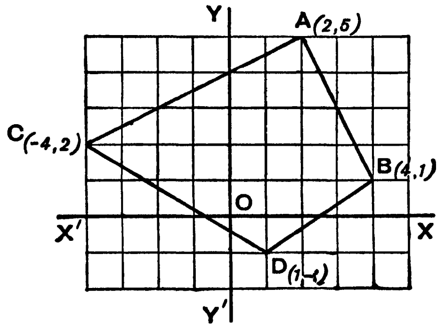

FIG. 21.

Otherwise, area of quadrilateral

$= \triangle ACB + \triangle CDB$

$= \frac{1}{2}(4-4+20+20-8-2)+\frac{1}{2}(4+1+8-2+4+4)$

$= \frac{1}{2}(30)+\frac{1}{2}(19)$

$= \frac{49}{2}$.

EXERCISES

1. Find the area of the triangles with vertices the origin and

(i) (4, 6), (2, 6).

(ii) (2, -6), (8, 7).

(iii) (-5, -3), (-6, -8).

(iv) (4, -12), (3, -7).

(v) $(\frac{3}{2}, \frac{2}{3})$, $(3, \frac{8}{3})$.

(vi) (a, b), (b, a).

(vii) $\left(\frac{a}{b}, \frac{a}{c}\right)$, $\left(\frac{b}{a}, \frac{b}{c}\right)$.

(viii) $(a \cos \alpha, a \sin \alpha)$, $(b \cos \beta, b \sin \beta)$.

(ix) $(\cos \theta, \sin \theta)$, $(\cos \overline{\theta + \alpha}, \sin \overline{\theta + \alpha})$.

(x) $(a \cos \overline{\alpha - \theta}, a \sin \overline{\alpha - \theta})$, $(b \cos \alpha, b \sin \alpha)$.

2. A, B, C, D are the points $(\frac{1}{3}, -\frac{1}{3})$, $(3, 3)$, $(2-\sqrt{3}, \sqrt{3}-2)$, $(2+\sqrt{3}, 2+\sqrt{3})$ and O is the origin; show that $\triangle OAB = \triangle OCD$.

3. A, B, C, D are the points $(3, 1)$, $(2, 4)$, $(2, 2)$, $(3, z)$ and O is the origin; $\triangle OAB = \triangle OCD$ in magnitude and sign; find z.

4. A, B, C, D are the points $(z-4, -2)$, $(z, z+3)$, $(2z+1, 1)$, $(z-3, 1)$ and O is the origin; find the value of the ratio $\triangle OAB : \triangle OCD$ and hence show that the areas of the triangles are equal in magnitude and sign when $z=4$.

5. A, B, C, D are the points $(3, 1)$, $(2, 4)$, $(2, 2)$, $(3-2z, z^2+5)$ and O is the origin; if $\triangle OAB = \triangle OCD$ in magnitude and sign, find z.

6. $\triangle OAB$ has vertices $(0, 0)$, $(b \cos \alpha, -b \sin \alpha)$, $(\sin \beta, \cos \beta)$; show that the area of $\triangle OAB$ is maximum when $\alpha = \beta$, and find the maximum area.

7. Find the area of the triangles with vertices

(i) $(-1, -3)$, $(4, 1)$, $(3, 5)$.

(ii) $(1, 3)$, $(3, 4)$, $(2, 6)$.

(iii) $(-4, -5)$, $(-1, -4)$, $(-3, -1)$.

(iv) $(\frac{3}{2}, -4)$, $(\frac{5}{2}, 3)$, $(-\frac{3}{2}, 2)$.

(v) $(-4, 5)$, $(3, -2)$, $(0, -4)$.

(vi) $(6, \frac{1}{2})$, $(-4, \frac{5}{2})$, $(-6, -\frac{7}{2})$.

8. A, B, C are the points $(-1, 3)$, $(2, -4)$, $(3, 2)$ relative to axes inclined at 30°; find the area of $\triangle ABC$.

9. Show that the area of the triangle with vertices $(t, t-2)$, $(t+3, t)$, $(t+2, t+2)$ is independent of t.

10. Find the area of the quadrilaterals with vertices

(i) $(0, 0)$, $(4, 6)$, $(2, 5)$, $(5, 1)$.

(ii) $(3, 3)$, $(1, 4)$, $(-2, 1)$, $(2, -3)$.

(iii) $(4, \frac{1}{2})$, $(\frac{5}{2}, 3)$, $(-4, -\frac{3}{2})$, $(1, -6)$.

(iv) $(-1, -1)$, $(-4, -2)$, $(-5, -4)$, $(-2, -3)$.

11. Show that the following sets of points are collinear:

(i) $(0, 1)$, $(2, 5)$, $(4, 9)$.

(ii) $(3, 0)$, $(2, \frac{1}{2})$, $(4, -\frac{1}{2})$.

(iii) $(4, \frac{5}{2})$, $(-2, -2)$, $(\frac{1}{2}, -\frac{1}{8})$.

(iv) $(2, a+2)$, $(-2, a-2)$, $(3, a+3)$.

(v) $\left(-2, -\frac{1}{b}\right)$, $\left(-3, -\frac{2}{b}\right)$, $\left(1, \frac{2}{b}\right)$, (b^2-1, b).

(vi) $(a, 0)$, $(am^2, 2am)$, $\left(\frac{a}{m^2}, -\frac{2a}{m}\right)$.

12. Find the area of the triangle with vertices $(2, -1)$, $(a+1, a-3)$, $(a+2, a)$ and show that these points are collinear when $a=\frac{1}{2}$.

13. Find the area of the triangle with vertices $(p+1, 1)$, $(2p+1, 3)$, $(2p+2, 2p)$ and show that these points are collinear when $p=2$ or $-\frac{1}{2}$.

14. The points $(2, \frac{3}{2})$, $(-3, -\frac{7}{2})$, $(z, \frac{9}{2})$ are collinear; find z.

15. A, B, C are the points $(3, 4)$, $(5, 2)$, (x, y); if area of $\triangle ABC$ is $+3$, show that $x+y-10=0$.

16. A, B, C are the points (x, y), $(-3, 2)$, $(-4, -4)$; if area of $\triangle ABC$ is $+\frac{35}{2}$, show that $6x-y-15=0$.

17. The vertices of a quadrilateral, in order, are $(-2, 3)$, $(-3, -2)$, $(2, -1)$, (x, y); if its area is $+14$, show that $x+y-2=0$.

CHAPTER IV

THE STRAIGHT LINE

§ 15. *Equation of the Locus of a Point.*

Definition. The equation of the locus of a point is the equation which is satisfied by the coordinates of every point lying on the locus, and by the coordinates of no other point; e.g.:

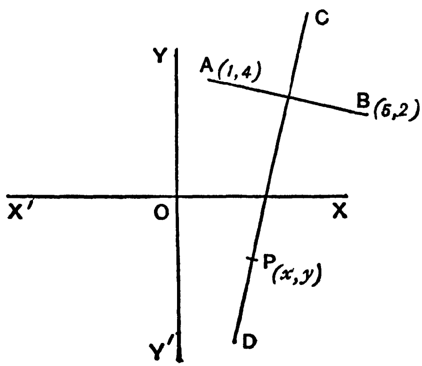

Fig. 22.

If $P(x, y)$ (Fig. 22) is any point equidistant from $A(1, 4)$ and $B(5, 2)$, we have

$$PA^2 = PB^2$$

and $\therefore$ $$(x-1)^2 + (y-4)^2 = (x-5)^2 + (y-2)^2,$$

i.e. $$8x - 4y = 12,$$

i.e. $$2x - y = 3. \quad\text{..............................(i)}$$

Moreover, if $P(x, y)$ is any point whose coordinates satisfy equation (i), we have

$$8x - 4y = 12,$$

and $\therefore$ $$(x-1)^2 + (y-4)^2 = (x-5)^2 + (y-2)^2,$$

i.e. $$PA^2 = PB^2,$$

i.e. P is equidistant from A and B.

Hence the equation, $2x - y = 3$, is the equation of the locus of points equidistant from A and B.

As this locus is CD, the right-bisector of AB, we say that equation (i) is "the equation of the line CD" or that it "represents the line CD," while CD is called briefly "the line $2x - y = 3$."

Example 1. *O and A are the origin and the point* (3, 4) *respectively* ; *the area of* $\triangle OAB$, *the vertices being taken in the order named, is* 7 ; *find the equation of the locus of B.*

Let B be the point (x, y) ;

then area of $\triangle OAB = \frac{1}{2}(3y - 4x)$

$$\therefore \frac{1}{2}(3y - 4x) = 7,$$

i.e. $$4x - 3y + 14 = 0,$$

which is the required equation.

Example 2. *Find the equation of the circle whose centre is the origin O and whose radius is a.*

Let $P(x, y)$ be any point on the circle ;

then $$OP^2 = x^2 + y^2$$

$$\therefore x^2 + y^2 = a^2,$$

which is the required equation.

Example 3. *P is a variable point* $(a+3, 2a-1)$ *where a may have any value* ; *find the equation of the locus of P, show that the locus passes through the point* (4, 1) *and determine the points at which it cuts the coordinate axes.*

Let the point (x, y) be any point on the locus:

then $$x = a + 3$$

and $$y = 2a - 1$$

$$\therefore \; x - 3 = \frac{y+1}{2},$$

i.e. $$2x - y - 7 = 0, \qquad \text{(i)}$$

which is the required equation.

Equation (i) is satisfied when $x = 4$ and $y = 1$;

$\therefore$ the point $(4, 1)$ lies on the locus.

Let the locus cut the x- and y-axes at A, B respectively;

then at A, $y = 0$ and $\therefore$ from equation (i) $x = \frac{7}{2}$;

and at B, $x = 0$ and $\therefore$ from equation (i) $y = -7$

$\therefore$ A, B are the points $(\frac{7}{2}, 0)$, $(0, -7)$ respectively.

Exercises

1. Find the equation of the locus of a point P such that

(i) it is equidistant from the coordinate axes;

(ii) the sum of its distances from the axes is 6;

(iii) the ratio, distance from x-axis : distance from y-axis $= 5 : 4$;

(iv) the sum of the squares of its distances from the axes is 25;

(v) the ratio, square of distance from x-axis : square of distance from y-axis $= 4 : 1$.

2. The point P is equidistant from the origin and the point $(4, 4)$; show that the equation of its locus is $x + y = 4$, and that the locus cuts the coordinate axes at points equidistant from the origin.

3. The point $P(x, y)$ moves so that $PA^2 - PB^2 = 5$, where A, B are the points $(-2, 3)$, $(3, 4)$; show that the locus of P has equation

$$10x + 2y = 17.$$

4. The point $P(x, y)$ moves so that $PA^2 + PB^2 = 44$, where A, B are the points $(-3, 2)$, $(3, -2)$; show that the locus of P has equation

$$x^2 + y^2 = 9.$$

5. The point P is equidistant from the points $(-2, 5)$ and $(2, 9)$; the point Q is equidistant from the points $(4, 1)$ and $(6, 5)$; find the equations of the loci of P and Q and show that the point $(3, 4)$ lies on each locus.

6. The vertices of a triangle are $A(x, y)$, $B(-4, -2)$ and $C(4, 2)$; if the median from A to BC is of constant length 5, show that the locus of A has equation

$$x^2+y^2=25.$$

7. O, A, B, C are the points (0, 0), (3, 5), (2, 6), (x, y), B and C being on the same side of OA; $\triangle OAC=2\triangle OAB$; show that the locus of C has equation

$$5x-3y+16=0.$$

8. The point $P(x, y)$ is equidistant from the line $x=-a$ and the point $(a, 0)$; show that its locus has equation

$$y^2=4ax.$$

9. P is a variable point $\left(t+\frac{1}{t}, t-\frac{1}{t}\right)$; show that the locus of P has equation

$$x^2-y^2=4.$$

10. P is a variable point $(at^2, 2at)$ where t may have any value; show that the locus of P has equation

$$y^2=4ax.$$

11. P is a variable point $(2t^2-1, t+1)$; show that the locus of P has equation

$$2y^2-x-4y+1=0.$$

12. P is a variable point $(a\cos\theta, b\sin\theta)$ where θ may have any value; show that the locus of P has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

13. P is a variable point $\left(a\cos\theta, b\sin\frac{\theta}{2}\right)$, where θ may have any value; show that the locus of P has equation

$$\frac{2y^2}{b^2}+\frac{x}{a}-1=0.$$

14. P is a variable point $(2t^2+t-1, t-1)$; show that the locus of P has equation

$$2y^2-x+5y+2=0$$

and determine the points at which the locus meets the y-axis.

15. A, B, C are the points $(-2, 1)$, $(2, -3)$, $(4, -2)$ relative to axes inclined at 30°; P is such that the area $ABCP$ is $+6$; show that the locus of P has equation

$$x+2y-4=0.$$

16. A variable point P is equidistant from the points (1, 4), (5, 2) relative to axes inclined at 60°; show that the locus of P has equation

$$x=3.$$

§ **16.** *Lines parallel to the coordinate axes.*

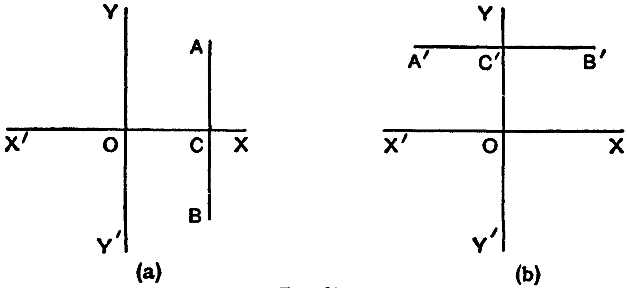

FIG. 23.

Let AB (Fig. 23 (a)), parallel to YY' cut $X'X$ at C, where $OC = a$.

The abscissa of any point on AB is OC, i.e. at any point on AB,

$$x = a,$$

which is therefore the equation of AB.

Similarly if $A'B'$ (Fig. 23 (b)), parallel to $X'X$, cuts YY' at C' where $OC' = b$, the equation of $A'B'$ is

$$y = b.$$

Note. The equation of the y-axis is $x = 0$; that of the x-axis is $y = 0$

§ **17.** *Tangent Form of the equation of a Straight Line.*

Let AB (Fig. 24), a line of gradient m, cut YY' at C, where $OC = c$.

Then, if P (x, y) is any point on AB,

$$\text{gradient of } AB = \frac{y-c}{x}$$

$$\therefore \frac{y-c}{x} = m,$$

i.e. $$y = mx + c,$$

which is therefore the equation of AB.

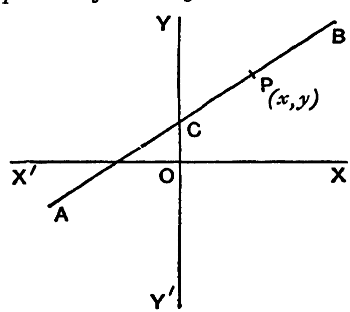

FIG. 24.

Note 1. This form of the equation of a line is called the tangent form, because m is the tangent of the angle of inclination of AB to the x-axis.

Note 2. The line AB is said to make an intercept OC or c on the y-axis; the intercept may be +ive or −ive.

Note 3. If the line AB passes through the origin, $c=0$, and the equation of AB is

$$y=mx.$$

Note 4. Let (r, θ) be the polar coordinates of any point on a straight line passing through the origin and inclined at angle α to the x-axis; then

$$\theta=\alpha,$$

which is ∴ the polar equation of the line.

Note 5. In any particular case the equation $y=mx+c$ may be derived as follows:

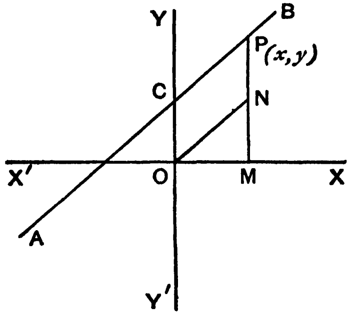

FIG. 25.

Let ON (Fig. 25), parallel to AB, meet MP, the ordinate of $P(x, y)$, at N.

Then

$$\begin{aligned} y &= MP \\ &= MN+NP \\ &= \frac{MN}{OM}\cdot OM+OC, \end{aligned}$$

i.e.

$$y=mx+c.$$

Note 6. If the axes are inclined at angle ω,

$$\text{gradient of line} = \frac{(y-c)\sin\omega}{x+(y-c)\cos\omega};$$

$\therefore$ if θ is the inclination of the line to the x-axis,

$$\frac{(y-c)\sin\omega}{x+(y-c)\cos\omega}=\frac{\sin\theta}{\cos\theta},$$

i.e. $$(y-c)\sin(\omega-\theta)=x\sin\theta,$$

i.e. $$y=\frac{\sin\theta}{\sin(\omega-\theta)}\cdot x+c.$$

Hence the equation of the line is of the form

$$y=mx+c,$$

but now $$m=\frac{\sin\theta}{\sin(\omega-\theta)}.$$

Example 1. *Find the equation of the straight line inclined at* 135° *to the x-axis, and meeting the y-axis at the point* (0, -4).

We have $$m=\tan 135°=-1$$

and $$c=-4\,;$$

$$\therefore \text{ the equation is } y=-x-4,$$

i.e. $$x+y+4=0.$$

Example 2. *A straight line has equation*

$$x-2y+6=0\,;$$

find its gradient, the intercept it makes on the y-axis, and the angle at which it is inclined to the line

$$x+3y=0.$$

Equation of first line can be written

$$y=\tfrac{1}{2}x+3\,;$$

$\therefore$ line has gradient $\frac{1}{2}$ and makes intercept 3 on the y-axis.

$$\text{Gradient of second line}=-\tfrac{1}{3}$$

$$\therefore \text{ tan of angle between lines}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}}=1$$

$\therefore$ the acute angle between the lines is 45°.

§ **18.** *Intercept Form of the equation of a Straight Line.*

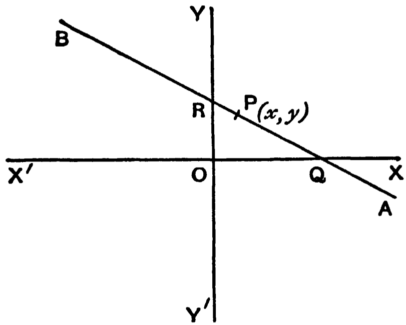

FIG. 26.

Let $P(x, y)$ (Fig. 26) be any point on AB; let AB cut the x-axis at Q, and the y-axis at R, where

$$OQ = a \quad \text{and} \quad OR = b.$$

Since $(a, 0)$ and (x, y) are points on AB,

$$\text{gradient of } AB = \frac{y}{x-a}.$$

Similarly
$$\text{gradient of } AB = \frac{y-b}{x}$$

$$\therefore \quad \frac{y}{x-a} = \frac{y-b}{x}$$

i.e.
$$bx + ay = ab$$

$$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1, \qquad \text{.......................(i)}$$

which is therefore the equation of AB.

Note 1. Equation (i) is sometimes written in the form

$$lx + my = 1,$$

where $l = \frac{1}{a}$, $m = \frac{1}{b}$.

Note 2. The Intercept Form may also be derived from the Tangent Form of the equation of a straight line.

AB (Fig. 26) has equation of the form

$$y = mx + b,$$

but AB passes through $Q(a, 0)$;

$$\therefore\ 0 = ma + b$$

$$\therefore\ m = -\frac{b}{a}$$

$$\therefore\ AB \text{ has equation } \quad y = -\frac{b}{a} \cdot x + b$$

i.e.
$$\frac{x}{a} + \frac{y}{b} = 1.$$

Note 3. In the case of oblique axes, the line has equation of the form, $y = mx + c$, and therefore the proof of Note 2 holds.

Example 1. *Find the equation of the straight line joining the points* (4, 0) *and* (0, -3).

Intercept on x-axis $= 4$

Intercept on y-axis $= -3$

$$\therefore\ \text{equation is } \frac{x}{4} - \frac{y}{3} = 1,$$

i.e.
$$3x - 4y = 12.$$

Example 2. *A straight line passes through the point* (3, 4) *and makes on the axes intercepts which are equal in magnitude and sign; find the equation of the line, and the length of the intercepts which it makes on the axes.*

Let each intercept $= a$

$$\therefore\ \text{equation is } \frac{x}{a} + \frac{y}{a} = 1$$

and since the point (3, 4) lies on the line

$$\therefore\ \frac{3}{a} + \frac{4}{a} = 1$$

$\therefore\ a = 7$ i.e. each intercept $= 7$,

and the equation is
$$\frac{x}{7} + \frac{y}{7} = 1,$$

i.e.
$$x + y = 7.$$

EXERCISES

1. Write down the equations of the lines determined by the following data :

(i) gradient $=\frac{3}{4}$, y-intercept $=\frac{1}{2}$.

(ii) gradient $=-\frac{5}{8}$, y-intercept $=-2$.

(iii) inclination to x-axis $=135°$, x-intercept $=-3$.

(iv) x-intercept $=3$, y-intercept $=-5$.

(v) x-intercept $=-\frac{1}{2}$, y-intercept $=-\frac{3}{4}$.

(vi) x-intercept $=\dfrac{p}{\cos\alpha}$, y-intercept $=\dfrac{p}{\sin\alpha}$.

(vii) x-intercept $=-\dfrac{c}{m}$, y-intercept $=c$.

2. Show that the following pairs of lines are parallel and state the length of the intercepts which each line makes on the x- and y-axes :

(i) $3x-2y+6=0$, $\quad 9x-6y-4=0$.

(ii) $5x+2y+10=0$, $\quad 15x+6y+20=0$.

(iii) $ax+by+c=0$, $\quad y=\dfrac{a}{b}(1-x)$.

(iv) $x\cos\alpha+y\sin\alpha=p$, $\quad y=-x\cot\alpha+a\operatorname{cosec}\alpha$.

3. Find the equation of the straight line which has gradient m and which passes through the point $\left(\dfrac{a}{m^2}, \dfrac{2a}{m}\right)$.

4. Find the equation of the line through the point $(a\cos\theta, a\sin\theta)$ making an x-intercept of $\dfrac{a}{\cos\theta}$.

5. A, B are points on the x-axis, C, D points on the y-axis ; the equations of AC, BD are respectively

$$4x+3y+6=0$$

and

$$x+2y-1=0\,;$$

show that $AB=CD$.

6. Find the equation of the line through the point $(3, -5)$, meeting the x- and y-axes in A and B respectively, where $4OA=OB$.

7. Find the equation of the line through the point $(-2, -5)$, meeting the x- and y-axes in A and B respectively, where $OA+2OB=0$.

8. P is the point $(t+5, 2t-4)$ where t may have any value ; find the equation of the locus of P and show that the locus makes x- and y-intercepts of 7 and -14 respectively.

9. The straight line $x+ay=a$ cuts the x- and y-axes at A and B respectively ; if $OA=3OB$, find the equation of AC where C is the point $(0, -9)$ and show that AC is perpendicular to AB.

10. Show that the locus of the variable point $(4t+1, 3t-1)$ is a straight line, parallel to the line joining the points (2, 5) and (6, 8).

11. The lines $8x-2y-3=0$ and $ax-5y+2=0$ are inclined at 45° ; find a.

12. Show that the lines

$$(1-m)y-(1+m)x=c$$

and

$$y=mx+c$$

are inclined to each other at 45°.

13. Find the gradient of the line

$$4x+5y-2=0,$$

the axes being inclined at the acute angle whose tangent is $\frac{3}{4}$.

14. A line referred to axes inclined at angle ω has equation

$$y=mx+c\,;$$

show that the line has gradient

$$\frac{m\sin\omega}{1+m\cos\omega}.$$

15. XOY is a right angle and XOY' is an acute angle whose tangent is $\frac{3}{4}$; show that the line whose equation is

$$x+2y-3=0,$$

with reference to axes OX, OY, has the same equation when referred to OX, OY' as axes.

§ **19.** *The equation of the straight line of gradient m passing through the point (x_1, y_1).*

If (x, y) be any point on the line,

$$\text{gradient of line} = \frac{y-y_1}{x-x_1}$$

$$\therefore\ \frac{y-y_1}{x-x_1}=m,$$

i.e.

$$y-y_1=m(x-x_1),$$

which is therefore the equation of the line.

Note. The reader should employ the method of § 17, Note 6, to show that when the axes are inclined at angle ω, the line passing through the point (x_1, y_1) and inclined at angle θ to the x-axis has equation

$$y - y_1 = \frac{\sin \theta}{\sin(\omega - \theta)}(x - x_1).$$

§ **20.** *The equation of the straight line passing through the points* (x_1, y_1) *and* (x_2, y_2).

Let (x, y) be any point on the line,

then $$\text{gradient of line} = \frac{y - y_1}{x - x_1}$$

and also $$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which is therefore the equation of the line.

Note 1. The coordinates x_1, y_1 are those of a fixed point; the coordinates x, y are those of any point on the line, and are referred to as "current coordinates."

Note 2. This equation holds for oblique axes, for each member of the equation $= \dfrac{\sin \theta}{\sin(\omega - \theta)}$, where ω is the angle between the axes and θ is the inclination of the line to the x-axis.

Example 1. *Find the equation of the line which has gradient* $\frac{3}{4}$ *and which passes through the point* (2, 1).

Equation is $$y - 1 = \tfrac{3}{4}(x - 2),$$
i.e. $$3x - 4y - 2 = 0.$$

Example 2. *Find the equation of the line joining the points* (1, 2), $(-2, 0)$ *and show that the point* $(-5, -2)$ *lies on the line.*

Equation is $$\frac{y - 2}{x - 1} = \frac{0 - 2}{-2 - 1}$$
$$= \tfrac{2}{3},$$
i.e. $$2x - 3y + 4 = 0.$$
Also $$2(-5) - 3(-2) + 4 = 0$$
$\therefore$ point $(-5, -2)$ lies on the line.

§ **21.** *The equation of the straight line passing through the point* (x_1, y_1) *and inclined at angle* θ *to the x-axis.*

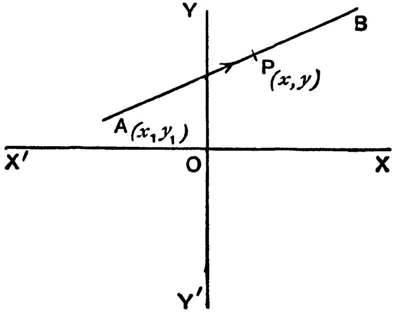

FIG. 27

Let AB (Fig. 27) be the line, and let $P(x, y)$ be a point on AB or AB produced, such that $AP = r$.

Then $$x - x_1 = r\cos\theta$$

and $$y - y_1 = r\sin\theta$$

$$\therefore \frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r, \qquad \text{.....................(i)}$$

which is therefore the equation of the line.

If P is a point on BA produced, equation (i) still holds, provided r is now considered to be negative.

Note 1. In the case of oblique axes, inclined at angle ω, the corresponding equation is,

$$\frac{x - x_1}{\frac{\sin(\omega - \theta)}{\sin\omega}} = \frac{y - y_1}{\frac{\sin\theta}{\sin\omega}} = r.$$

Note 2. Equation (i) is very useful in more advanced portions of analytical geometry.

Example 1. *A line of gradient* $\frac{3}{4}$ *passes through* $P(-2, -5)$; *find the coordinates of a point* Q *on the line, where* $PQ = 10$.

Let Q be the point (x, y).

Using the relation $\quad \dfrac{x-x_1}{\cos\theta}=r,$

i.e. $\quad x=x_1+r\cos\theta$

we have $\quad x=-2+10(\pm\tfrac{4}{5})$

$\quad =6 \text{ or } -10,$

and similarly $\quad y=-5+10(\pm\tfrac{3}{5})$

$\quad =1 \text{ or } -11$

$\therefore\ Q$ is the point $(6, 1)$ or $(-10, -11)$.

Example 2. *A is the point* (2, 1) *and B lies on the line*

$$x-y-2=0\,;$$

AB is inclined to the x-axis at the acute angle whose tangent is $\tfrac{3}{4}$; *find the length of AB.*

Let B be the point (x, y) and let $AB=r$.

Then $\quad x=2+\tfrac{4}{5}r,$

and $\quad y=1+\tfrac{3}{5}r\,;$

$$\therefore\ (2+\tfrac{4}{5}r)-(1+\tfrac{3}{5}r)-2=0\,;$$

$$\therefore\ \frac{r}{5}=1$$

$$\therefore\ r=5,$$

i.e. $\quad AB=5.$

Exercises

1. Write down the equations of the straight lines :

(i) of gradient $\tfrac{5}{6}$, and through the point $(-3, -1)$.

(ii) of gradient $-\tfrac{4}{5}$, and through the point $(-2, 4)$.

(iii) inclined at $60°$ to the x-axis, and through the point $(4, -8)$.

(iv) inclined at $135°$ to the x-axis, and through the point $(5, -2)$.

(v) inclined to the x-axis at the angle whose tangent is $\tfrac{3}{4}$, and through the point $(2, 6)$.

(vi) of gradient $-\dfrac{b}{a}$, and through the point $\left(b,\ b\,\overline{1-\dfrac{b}{a}}\right)$.

(vii) of gradient $2m$, and through the point $\left(\dfrac{a}{4m^2},\ \dfrac{a}{m}\right)$.

(viii) of gradient m, and through the point $(0,\ a\sqrt{1+m^2})$.

2. Write down the equations of the lines joining the following pairs of points:

(i) $(3, 8), (6, 12)$.

(ii) $(-1, -3), (6, 11)$.

(iii) $(2, -5), (0, -8)$.

(iv) $(\frac{1}{2}, \frac{3}{2}), (\frac{7}{2}, -\frac{5}{2})$.

(v) $(a, b), (b, a)$.

(vi) $(m, m^2+c), \left(\frac{m-c}{m}, m\right)$.

(vii) $\left(\frac{a}{m^2}, \frac{2a}{m}\right), \left(\frac{a}{4m^2}, \frac{a}{m}\right)$.

(viii) $\left(ct, \frac{c}{t}\right), \left(2ct, \frac{c}{2t}\right)$.

3. A, B, C, D are the points $(-1, 8), (4, 18), (-2, 4), (6, 6)$; find the equation of the line bisecting CD and perpendicular to AB.

4. Find the equations of the sides of the triangle with vertices $(3, 6), (-2, 4)$ and $(-1, -3)$.

5. A, B are the points $(3, 7), (-11, -1)$; find the equation of the line through the point $(4, -6)$ and bisecting AB.

6. A, B, C are the points $(-1, 2), (8, 5), (4, 9)$; D lies on AB and $AD : DB = 1 : 2$; find the equation of DC.

7. A, B, C are the points $(3, 4), (-5, 0), (-1, 12)$; find the equations of AB and AC, and show that AB is perpendicular to AC.

8. $P(a, b)$ lies on the line

$$6x - y = 1,$$

and $Q(b, a)$ lies on the line

$$2x - 5y = 5;$$

find the equation of PQ.

9. A, B, C are the points

$$\left(-\frac{a}{m^2}, 0\right), \left(\frac{a}{m^2}, \frac{2a}{m}\right), \left(\frac{a}{m^2}, -\frac{2a}{m}\right);$$

find the equations of AB and AC, and show that AB is perpendicular to AC when $m = 1$.

10. Find the points on the following lines at which the x- and y-coordinates are equal, and hence write each equation in the form

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\alpha}{\sin\theta} = r;$$

(i) $4x - 3y - 1 = 0$; (ii) $5x + 12y + 17 = 0$; (iii) $x - \sqrt{3}\,y = 0$.

11. A line of gradient $\frac{5}{12}$ passes through $P(3, -\frac{7}{2})$; find the coordinates of a point Q on the line where $PQ = \frac{13}{2}$.

12. A is the point $(-5, -3)$ and B lies on the line

$$x - 3y - 1 = 0;$$

AB is inclined to the x-axis at the acute angle whose tangent is $\frac{5}{12}$; find the length of AB.

13. The point $P(-2, -3)$ lies on the line AB whose equation is

$$4x+ay=1\ ;$$

find the coordinates of A and B, given that $AP=PB=10$.

14. Find the equation of the line of gradient 2, passing through the point $(2, -3)$, the axes being inclined at the acute angle whose tangent is $\frac{4}{3}$.

§ **22.** *The General Linear Equation.*

The equation

$$Ax+By+C=0, \qquad \text{(i)}$$

where A, B, C are constants, represents a straight line.

If $B \neq 0$, (i) can be written,

$$y=-\frac{A}{B}x-\frac{C}{B},$$

and $\therefore$ represents a straight line of gradient $-\frac{A}{B}$ and passing through the point $\left(0, -\frac{C}{B}\right)$.

If $B=0$, (i) can be written

$$x=-\frac{C}{A},$$

and $\therefore$ represents a straight line parallel to the y-axis.

Note 1. The same proof holds in the case of oblique axes, except that $-\frac{A}{B}$ is no longer the gradient of the line.

Note 2. The line (i) is parallel to the x-axis if $A=0$, is parallel to the y-axis if $B=0$, and passes through the origin if $C=0$.

Note 3. Equation (i) can be written

$$-\frac{A}{C}x-\frac{B}{C}y=1,$$

i.e. in the form

$$lx+my=1 \qquad \text{(ii)}$$

$\therefore$ any equation of this form represents a straight line.

When expressed in polar coordinates, equation (ii) becomes

$$l\cos\theta+m\sin\theta=\frac{1}{r}.$$

Note 4. The equation of a straight line is sometimes given in the form

$$x = a + bt, \quad \text{.........................(i)}$$
$$y = c + dt, \quad \text{.........................(ii)}$$

where a, b, c, d are constants and t is a variable, and where (i) and (ii) are called the freedom equations of the straight line. The reader may verify, by eliminating t, that (i) and (ii) represent a straight line.

Example. The coordinates of a variable point P are $x = 3 + 4t$, $y = 2 - 6t$; show that the locus of P is a straight line.

$$x = 3 + 4t,$$
$$y = 2 - 6t$$

and $\therefore$ $$\frac{x-3}{4} = \frac{y-2}{-6},$$

i.e. $$3x + 2y - 13 = 0$$

$\therefore$ the locus of P is the straight line $3x + 2y - 13 = 0$.

§ **23.** *Angle between two lines.*

The lines $$A_1x + B_1y + C_1 = 0, \quad \text{.........................(i)}$$
$$A_2x + B_2y + C_2 = 0, \quad \text{.........................(ii)}$$

have gradients $-\dfrac{A_1}{B_1}$, $-\dfrac{A_2}{B_2}$

$\therefore$ the tangent of the angle between (i) and (ii)

$$= \pm \frac{-\dfrac{A_1}{B_1} + \dfrac{A_2}{B_2}}{1 + \dfrac{A_1A_2}{B_1B_2}}$$

$$= \mp \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2}. \quad \text{.........................(iii)}$$

Note 1. The conditions for parallelism and perpendicularity may be deduced from expression (iii); the lines are parallel or perpendicular according as the tangent of the angle between them is zero or infinite;

$\therefore$ the lines are parallel if $A_1B_2 = A_2B_1$,

i.e. if $$\frac{A_1}{B_1} = \frac{A_2}{B_2},$$

and the lines are perpendicular if $A_1A_2 + B_1B_2 = 0$.

The reader should compare § 10, Note 2.

Note 2. The reader may verify that if the axes are inclined at angle ω, the tangent of the angle between the lines is

$$\frac{(A_1B_2 - A_2B_1)\sin\omega}{A_1A_2 + B_1B_2 - (A_1B_2 + A_2B_1)\cos\omega}.$$

The condition for parallelism is $\therefore$ the same as in the case of rectangular axes, and the condition for perpendicularity is

$$A_1A_2 + B_1B_2 - (A_1B_2 + A_2B_1)\cos\omega = 0.$$

§ **24.** *The equation of the line passing through the point* (x_1, y_1) *and* (i) *parallel,* (ii) *perpendicular to the line* $Ax + By + C = 0$, *is*

$$\text{(i)}\quad Ax + By = Ax_1 + By_1,$$
$$\text{(ii)}\quad Bx - Ay = Bx_1 - Ay_1.$$

These equations are linear and are satisfied by the coordinates of the point (x_1, y_1) $\therefore$ represent straight lines passing through the point (x_1, y_1).

$$\text{Gradient of (i)} = -\frac{A}{B} \quad\text{and}\quad \text{gradient of (ii)} = \frac{B}{A};$$

$\therefore$ (i) is parallel to, and (ii) is perpendicular to the given line.

Note. Equation (i) also gives the parallel in the case of oblique axes.

Example. Find the equations of the lines passing through the point (2, 3) *and respectively parallel and perpendicular to the line*

$$2x - 3y = 1.$$

Equations are $\quad 2x - 3y = 4 - 9,$

i.e. $\quad 2x - 3y + 5 = 0$

and $\quad 3x + 2y = 6 + 6,$

i.e. $\quad 3x + 2y - 12 = 0.$

§ **25.** *Intersection of Two Lines.*

The coordinates of the point of intersection of two loci satisfy the equations of both loci and are therefore determined by

solving these equations simultaneously, e.g. the coordinates of the point of intersection of the lines

$$A_1x + B_1y + C_1 = 0, \quad \text{......................(i)}$$
$$A_2x + B_2y + C_2 = 0, \quad \text{......................(ii)}$$

satisfy both equations (i) and (ii), and hence the coordinates are the roots of equations (i) and (ii) solved simultaneously. The point of intersection is $\therefore$ the point

$$\left(\frac{B_1C_2 - B_2C_1}{A_1B_2 - A_2B_1}, \frac{C_1A_2 - C_2A_1}{A_1B_2 - A_2B_1}\right).$$

Note. An alternative method of finding the point of intersection of two lines is shown in Example 2.

Example 1. *Find the point of intersection of the lines*

$$2x - 3y + 4 = 0$$

and
$$x - 4y + 7 = 0.$$

The point is
$$\left(\frac{-21 + 16}{-8 + 3}, \frac{4 - 14}{-8 + 3}\right),$$

i.e.
$$(1, 2).$$

Example 2. *Find the point of intersection of the lines*

$$3x - y - 1 = 0, \quad \text{..............................(i)}$$
$$x + 2y - 5 = 0. \quad \text{..............................(ii)}$$

If in equation (i)
$$x = t,$$
$$y = 3t - 1$$

$\therefore$ for all values of t the point $(t, 3t - 1)$ lies on (i), but the point $(t, 3t - 1)$ lies on (ii) if

$$t + 2(3t - 1) - 5 = 0,$$

i.e. if
$$t = 1$$

$\therefore$ the point $(t, 3t - 1)$ when $t = 1$ is the point of intersection of (i) and (ii),

i.e. the point (1, 2) is the required point.

Exercises

1. The point P moves so that the square of its distance from the point (1, 2) exceeds by 3 the square of its distance from the point (3, 5); show that the locus of P is a straight line.

2. Show that for all values of t the point $\left(\frac{t+1}{t}, \frac{4t+1}{t}\right)$ lies on a straight line.

3. A point $P(x, y)$ moves so that it is always equidistant from the points $A(2, 3)$ and $B(-3, -2)$; find the equation of its locus and show analytically that it is the right bisector of AB.

4. Test each of the following pairs of lines for parallelism and perpendicularity, and, when the lines are neither parallel nor perpendicular, find the tangent of the angle between them:

(i) $2x-3y+1=0,\ 4x-6y+3=0.$
(ii) $x+2y+4=0,\ x-3y+2=0.$
(iii) $x+2y+3=0,\ 3x+6y+1=0.$
(iv) $3x-4y+12=0,\ 4x+3y-1=0.$
(v) $2x+3y-1=0,\ x-2y+3=0.$
(vi) $2x+3y+6=0,\ 6x-4y+3=0.$
(vii) $x+4y-3=0,\ 3x-2y+1=0.$
(viii) $3x+4y-7=0,\ 2x+5y+7=0.$

5. Find the equation of the line (i) through $(4, -2)$ parallel to the line

$$3x+4y+6=0$$

and (ii) through $(-10, -1)$ perpendicular to the line

$$5x-6y-2=0.$$

6. Find the equation of the line through the point $(-2, 6)$ perpendicular to the line

$$2x+3y-1=0,$$

and find where the two lines meet.

7. Find the equation of the line through the point

$$\left(\frac{a\cos\alpha}{\cos\theta}, \frac{a\sin\alpha}{\cos\theta}\right)$$

parallel to the line

$$x\cos(\alpha-\theta)+y\sin(\alpha-\theta)=p.$$

8. Find the equation of the line through the point $(a\sec\theta.\ b\tan\theta)$ perpendicular to the line

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

9. AB makes intercepts 3 and -2 on the x- and y-axes respectively: CD makes intercepts 12 and 4; find the equations of AB and CD, and the coordinates of their point of intersection.

10. The lines $$x+y=4$$
and $$x+2y=1$$
meet the line
$$4x+3y=14$$
at P and Q respectively; find the length of PQ.

11. AB passes through the point $(a, 0)$ and is inclined at 45° to the x-axis; CD passes through the point $(0, b)$ and is inclined at 135° to the x-axis; find the coordinates of the point of intersection of AB and CD.

12. A, B, C, D are the points (10, 0), (0, 5), (6, 0), (0, 9); AB and CD meet at E; find the equation of OE where O is the origin.

13. Show that the lines
$$x+2y-15=0,$$
$$2x-y-10=0,$$
$$2x+4y+5=0,$$
and $$2x-y=0$$
enclose a rectangle, and find the coordinates of the point of intersection of the diagonals.

14. Find the distance measured along the line
$$4x-3y+2=0,$$
from the point (1, 2) to the line
$$x-2y-2=0.$$

15. Find the circumcentre of the triangle with vertices $A(-3, 3)$, $B(-5, -1)$, $C(4, -4)$.

16. AD, BE, CF are altitudes of the triangle with vertices $A(0, 4)$, $B(-5, -4)$, $C(3, -4)$; show that BE and CF intersect on the y-axis, and hence show that the altitudes are concurrent.

17. Find the orthocentre of the triangle with vertices $(-2, -3)$, $(6, 1)$, $(1, 6)$.

18. Find the area of the triangle with sides
$$7x+8y-5=0,$$
$$11x+y+50=0,$$
$$4x-7y-26=0.$$

19. Find the centroid of the triangle with vertices $(3, -4)$, $(-5, 4)$, $(-3, -8)$.

20. Show that the lines

$$y = mx + \frac{a}{m}$$

and
$$y + \frac{x}{m} + am = 0,$$

meet at right angles on the line

$$x = -a.$$

21. A, B, C are the points

$$\left(-a,\, a\overline{\frac{1}{m} - m}\right),\ \left(\frac{a}{m^2},\ \frac{2a}{m}\right),\ (am^2,\ -2am);$$

find the equations of AB and AC, and show that they meet at right angles on the line

$$x = -a.$$

22. Show that for all values of m, the foot of the perpendicular from $(a, 0)$ to the line

$$y = mx + \frac{a}{m}$$

lies on the y-axis.

23. PQ makes intercepts $2a$ and a, RS makes intercepts $-a$ and $2a$ on the x- and y-axes respectively; show that PQ and RS intersect on the line

$$3x + y = 0.$$

24. Show that the lines

$$\begin{aligned} x &= t \qquad &\text{and}\qquad x &= 2t, \\ y &= 2t + 1 & y &= -t - 4, \end{aligned}$$

intersect at right angles at the point $(-2, -3)$.

25. Show that the lines

$$\begin{aligned} x &= 1 - 3t \qquad &\text{and}\qquad x &= 4 - 5t, \\ y &= 1 + t & y &= 2t - 1, \end{aligned}$$

intersect at the point $(-11, 5)$.

26. Find the point of intersection of the lines

$$\begin{aligned} x &= \frac{2t+1}{t+1} \qquad &\text{and}\qquad x &= \frac{1+t}{1-t}, \\ y &= \frac{t}{t+1} & y &= \frac{3t}{1-t}. \end{aligned}$$

27. Show that the lines

$$\begin{aligned} 2x &= 1 - 4t \qquad &\text{and}\qquad x &= 6 - 2t, \\ y &= 1 + t & y &= t - 1, \end{aligned}$$

are parallel.

28. The lines

$$x = \frac{at-3}{a} \quad \text{and} \quad x = 1 - 2bt,$$
$$y = 1 - at \qquad\qquad y = \frac{2(bt-2)}{b}$$

are parallel; show that $ab = 1$.

29. Find the area enclosed by the line

$$2x - 3y + 12 = 0$$

and the coordinate axes, the latter being inclined at 30°.

30. Find the area of the triangle with sides

$$2x + y - 10 = 0,$$
$$x - 3y + 2 = 0,$$
$$3x - 2y - 1 = 0,$$

the axes being inclined at 30°.

§ **26.** *The Perpendicular Form of the equation of a Straight Line.*

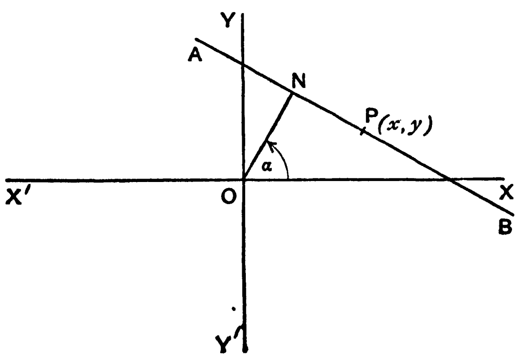

FIG. 28.

Let ON (Fig. 28) be the perpendicular from the origin to the line AB; let $ON = p$, let $X\hat{O}N = \alpha$, and let $P(x, y)$ be any point on AB.

N has polar coordinates (p, α) and $\therefore$ Cartesian coordinates $(p \cos \alpha, p \sin \alpha)$;

$$\therefore \text{ gradient of } AB = \frac{p \sin \alpha - y}{p \cos \alpha - x},$$

but the gradient of ON is $\tan\alpha$, and AB is perpendicular to ON

$$\therefore \text{ gradient of } AB = -\cot\alpha;$$

$$\therefore \frac{p\sin\alpha - y}{p\cos\alpha - x} = -\frac{\cos\alpha}{\sin\alpha},$$

i.e. $$x\cos\alpha + y\sin\alpha = p(\sin^2\alpha + \cos^2\alpha),$$

i.e. $$x\cos\alpha + y\sin\alpha = p,$$

which is therefore the equation of AB.

Note 1. The perpendicular p will be considered positive in all cases.

Note 2. In any particular case the perpendicular form of the equation may be derived as follows:

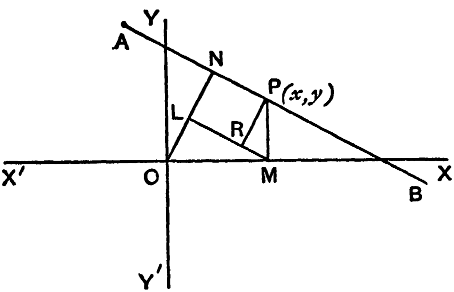

FIG. 29.

Let MP (Fig. 29) be the ordinate of the point P. Draw $ML \perp ON$ and $PR \perp ML$.

Then $$P\hat{M}R = \alpha;$$

$$\begin{aligned} \therefore\ ON \text{ which } &= OL + LN \\ &= OL + RP \\ &= OM\cos\alpha + MP\sin\alpha \end{aligned}$$

$$\therefore\ p = x\cos\alpha + y\sin\alpha.$$

Note 3. It is left to the reader to show that the polar equation of the line is

$$r\cos(\theta - \alpha) = p.$$

Note 4. In the case of oblique axes, if

$$X\hat{O}N=\alpha \text{ and } N\hat{O}Y=\beta \text{ (Fig. 30),}$$

then $$OQ=p\sec\alpha, \quad OR=p\sec\beta\,;$$

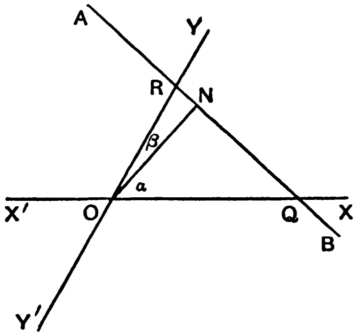

FIG 30.

∴ AB has equation

$$\frac{x}{p\sec\alpha}+\frac{y}{p\sec\beta}=1,$$

i.e. $$x\cos\alpha+y\cos\beta=p.$$

§ **27.** *Perpendicular Form of the General Linear Equation.*

We may change the equation $Ax+By+C=0$ to the perpendicular form as follows :

(i) C +ive. $\qquad Ax+By+C=0 \qquad$ ...(i)

$$\therefore\ -\frac{Ax}{\sqrt{A^2+B^2}}-\frac{By}{\sqrt{A^2+B^2}}=\frac{C}{\sqrt{A^2+B^2}},$$

i.e. $$x\cos\alpha+y\sin\alpha=\frac{C}{\sqrt{A^2+B^2}}, \text{ ... where } \tan\alpha=\frac{B}{A}. \quad \text{(ii)}$$

(ii) C −ive. $\qquad Ax+By+C=0$

$$\therefore\ \frac{Ax}{\sqrt{A^2+B^2}}+\frac{By}{\sqrt{A^2+B^2}}=\frac{-C}{\sqrt{A^2+B^2}},$$

i.e. $$x\cos\alpha+y\sin\alpha=\frac{-C}{\sqrt{A^2+B^2}}, \text{ ... where } \tan\alpha=\frac{B}{A}. \quad \text{(iii)}$$

Note 1. Equations (ii) and (iii) are arranged to give a +ive term on the right-hand side, since in the perpendicular form of the equation p is +ive.

Note 2. There are two angles α lying between 0 and 2π and such that $\tan \alpha = \frac{B}{A}$; the value of α is determined by the signs of $\sin \alpha$ and $\cos \alpha$.

Example. Show that the lines

$$4x + 3y - 25 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{i})$$

$$4x - 3y + 25 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{ii})$$

are tangents to the circle, of radius 5 *and centre the origin.*

Equation (i) in perpendicular form is

$$\frac{4x}{\sqrt{16+9}} + \frac{3y}{\sqrt{16+9}} = \frac{25}{\sqrt{16+9}} = 5,$$

i.e. the perpendicular from the origin to (i) is equal in length to the radius of circle

$$\therefore \text{ (i) is a tangent to the circle.}$$

Similarly for line (ii) whose equation in perpendicular form is

$$-\frac{4x}{5} + \frac{3y}{5} = 5.$$

Exercises

1. Write the following equations in perpendicular form :

(i) $3x + 4y - 5 = 0$. (iv) $8x + 15y + 34 = 0$.

(ii) $x - \sqrt{3}\,y + 4 = 0$. (v) $5x - 12y + 26 = 0$.

(iii) $x + y = 4$. (vi) $40x - 9y = 41$.

2. The lines

$$3x - 4y = a$$

and

$$5x + 12y = 2(a+3)$$

are equidistant from the origin ; find the positive value of a.

3. Derive the equation

$$x \cos \alpha + y \sin \alpha = p$$

from the intercept form of the equation of the line.

4. Show that the line

$$x \cos \alpha + y \sin \alpha = x_1 \cos \alpha + y_1 \sin \alpha$$

passes through the point (x_1, y_1) and is parallel to the line

$$x \cos \alpha + y \sin \alpha = p.$$

5. Show that the equation

$$x \cos \alpha + y \sin \alpha = p$$

can be written in the form

$$\frac{x - p \cos \alpha}{-\sin \alpha} = \frac{y - p \sin \alpha}{\cos \alpha} = r.$$

6. The lines

$$x \cos \alpha + y \sin \alpha = p,$$
$$x \cos \alpha_1 + y \sin \alpha_1 = p_1,$$

intersect on the line

$$y = x \tan \theta;$$

show that $$p : p_1 = \cos(\theta - \alpha) : \cos(\theta - \alpha_1).$$

7. The sides of a quadrilateral, taken in order, are

$$x \cos \alpha + y \sin \alpha = p \quad \text{(i)}$$
$$x \cos \alpha_1 + y \sin \alpha_1 = p_1 \quad \text{(ii)}$$

and the perpendiculars from the origin to (ii), (i); show that

$$\text{area of quadrilateral} = \frac{2pp_1 - (p^2 + p_1^2) \cos(\alpha_1 - \alpha)}{2 \sin(\alpha_1 - \alpha)}.$$

Exercises

1. Show that the line from the point (2, 5) perpendicular to the join of the points (17, -7) and (2, -1) passes through the origin.

2. A line makes on the x- and y-axes intercepts a and b such that $\frac{1}{a} + \frac{1}{b} = \frac{1}{4}$; show that the line passes through the point (4, 4).

3. Show that for the line

$$y = mx + \frac{1 - m}{6},$$

the sum of the reciprocals of the intercepts on the x- and y-axes is independent of m.

4. A line of gradient $\frac{3}{4}$ meets the x- and y-axes in A and B respectively; show that the perpendiculars to the axes at A and B intersect on the line

$$3x + 4y = 0.$$

5. PQ has gradient $\frac{3}{4}$ and passes through the point $(-8, -3)$; AB passes through the point $(2, 2)$ and meets the x- and y-axes in A and B respectively, where $OA = 2OB$; show that AB and PQ meet on the y-axis.

6. Show that the lines

$$3x + 4y + 10 = 0$$

and

$$5x + 12y - 26 = 0$$

are equidistant from the origin.

7. Show that the origin lies on a bisector of one of the angles between the lines

$$6x - 8y + 5 = 0 \quad \text{and} \quad 14x + 48y + 25 = 0.$$

8. Show that the area enclosed by the lines

$$x = a, \quad y = b, \quad y = mx$$

is

$$\frac{1}{2m}(b - ma)^2;$$

show that this area is the same for any two values of m whose product is $\frac{b^2}{a^2}$.

9. The line

$$\frac{x}{a} + \frac{y}{b} = 1$$

meets the x-axis at A, the y-axis at B; P divides in the ratio $1 : k$ the line joining the origin O to A; Q divides OB in the ratio $1 : l$; find the equation of PQ and show that it divides AB in the ratio

$$-k : l.$$

10. The line

$$x \cos \theta + y \sin 0 = p,$$

where θ is variable, cuts the x- and y-axes at A, B; show that the locus of the middle point of AB has equation

$$p^2(x^2 + y^2) = 4x^2y^2.$$

11. A, B are the points (x_1, y_1), (x_2, y_2) and P divides AB in the ratio $k : l$; P lies on the line

$$Ax + By + C = 0;$$

show that

$$\frac{k}{l} = -\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C}.$$

12. Determine the ratios in which the coordinate axes divide the line joining the points $(-1, -10)$, $(2, -4)$.

13. Show that the line

$$3x - 4y + 1 = 0$$

bisects the join of $(4, -3)$, $(-2, 5)$.

14. Determine the ratio in which the line

$$2x-3y+1=0$$

divides the join of $(7,\ -6)$, $(-3,\ 2)$.

15. Determine the ratio in which the line

$$3x-4y+1=0$$

divides the join of $(4,\ -3)$, $(5,\ -1)$.

16. Prove that, if a straight line cuts the sides BC, CA, AB of $\triangle ABC$ at D, E, F,

$$\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=-1.$$

17. A, B are the points $(4,\ 2)$, $(-3,\ 1)$; a point P moves so that PA^2-PB^2 is constant; show that the locus of P is a straight line, and find the ratio in which it divides AB when $PA^2-PB^2=6$.

18. Show that the angle between the lines

$$x-y+2=0,\quad 3x+y+4=0$$

is equal to the angle between the lines

$$x-4y+3=0,\quad 9x-2y+1=0.$$

19. PQ is the line

$$y=\frac{x}{t}+at$$

and R, S are the points $(at^2,\ 2at)$, $(a,\ 0)$; show that R lies on PQ and that RS and the x-axis are equally inclined to PQ.

20. Find the feet of the perpendiculars from the point $(-1,\ 3)$ to the lines

$$2x-y+1=0,$$
$$3x+y-6=0,$$
$$x-3y-2=0,$$

and show that they are collinear.

21. Find the area enclosed by the lines

$$x-2y+3=0,\ 4x-y-9=0$$

and the parallels to those lines through the point $(-2,\ -3)$.

22. Show that the equation

$$\frac{x-x_1}{\cos\alpha}=\frac{y-y_1}{\sin\alpha}=r$$

represents the line through $(x_1,\ y_1)$ perpendicular to the line

$$x\cos\alpha+y\sin\alpha=p\ \text{(i)};$$

hence show that the perpendicular distance from the point (x_1, y_1) to the line (i) is numerically

$$p - x_1 \cos \alpha - y_1 \sin \alpha.$$

23. Show that the lines

$$y = \frac{x}{t} + at, \quad y = \frac{x}{t_1} + at_1$$

intersect at the point

$$P(att_1, \ a\overline{t+t_1});$$

show also that if t_1 varies as t the locus of P has equation

$$y^2 = kx,$$

where k is a constant.

24. Find the coordinates of a point P which lies on the line

$$x - 3y - 2 = 0$$

and which is equidistant from the points (2, 3), (6, −5).

25. Show that the locus of the point of intersection of the lines

$$x \sin \theta - y(\cos \theta - 1) = a \sin \theta$$

and

$$x \sin \theta - y(\cos \theta + 1) = -a \sin \theta$$

has equation

$$x^2 + y^2 = a^2.$$

26. The line $y = b$ meets the lines

$$x + 2y = 2 \quad \text{and} \quad x + 2y = 4$$

in A and B respectively; P, Q are the points (−4, 0), (−2, 0); show that PB and QA bisect each other, and that the common mid-point lies on the line

$$x + 2y = 0.$$

27. The line $y = x + c$ meets the lines

$$x = 3 \quad \text{and} \quad x = 5$$

in A and B respectively; P, Q are the points (0, 2), (0, 4); find the equations of AP and BQ, and the point of intersection of these lines when $c = 1$.

28. The line $y = mx$ meets the lines

$$x = 4 \quad \text{and} \quad x = 8$$

at P and Q respectively; PR is a line parallel to the x-axis, QR is a line of gradient 1; show that for all values of m, R lies on the line

$$x + y - 8 = 0.$$

29. A, B, C are the points

$$\left(-a, \ a t\,\overline{1 - \frac{1}{t^2}}\right), \quad (at^2, 2at), \quad \left(\frac{a}{t^2}, \ -\frac{2a}{t}\right);$$

AB and AC meet the y-axis in P and Q respectively; show that AB is perpendicular to AC and that

$$QP=a\left(\frac{t^2+1}{t}\right).$$

30. A, B, C are the points $(-2, 1), (2, 3), (3, 1)$; P moves so that

$$\triangle PAB=2\triangle PAC$$

in magnitude and sign; find the locus of P, and the area enclosed by it and the lines AB, BC.

31. Find the length of the line having gradient 3 and terminated by the point $(-1, 2)$ and the line

$$x-2y-5=0.$$

32. Find the equations of the lines of length 5, drawn from the point $(3, 1)$ to the line

$$x+y+3=0.$$

33. Find the length of the line of gradient $\frac{1}{2}$ and terminated by the point $P(3, 2)$ and the line

$$x+2y+3=0 \quad \text{(i)};$$

find also the equation of the other line having the same length and terminated by P and (i).

34. A straight line passing through $P(1, 2)$ is terminated by the coordinate axes; P divides the line internally in the ratio $2:1$; show that the line has equation

$$x+y=3 \quad \text{or} \quad 4x+y-6=0.$$

35. Show that the variable point $P(t-1, 2t-1)$ lies on the line

$$2x-y+1=0;$$

if PQ is the perpendicular from P to the line

$$x-3y-2=0,$$

show that the locus of the mid-point of PQ is the line

$$y=x.$$

36. Show that the line

$$x-2y+5=0$$

bisects all lines from the point $(-3, 6)$ to the line

$$x-2y-5=0.$$

37. P is a variable point $(2p-1, p+1)$; the line through P and having gradient 2 meets the line

$$x+y=0$$

at Q; R divides PQ externally in the ratio $3:1$; find the locus of R.

38. Prove that the line

$$2x+3y-20=0$$

passes through a point of trisection of all lines drawn from the point (6, 7) to the line

$$2x+3y+6=0.$$

39. A line through $P(2, -3)$ meets the lines

$$x-2y+7=0, \quad x+3y-3=0$$

at A, B respectively; P divides AB externally in the ratio $3:2$; show that AB has equation

$$2x+y-1=0.$$

40. Find the equation of the line terminated by the lines

$$2x+y-3=0, \quad 3x-2y+1=0$$

and bisected at the point (2, 3).

41. Find the gradient of the line whose mid-point is $(t-3, t+4)$ and which is terminated by the lines

$$2x+y-2=0, \quad x+2y-3=0.$$

42. P is the point $(-1, 2)$; a variable line through P cuts the x- and y-axes at A, B respectively; Q is a point on AB such that PA, PQ, PB are in Harmonic Progression; show analytically that the locus of Q is the line

$$y=2x.$$

43. The axes being inclined at 60°, find the equation of the right bisector of the line joining the points $(-1, 4)$, $(3, -2)$.

44. A, B are the points (2, 7), (1, 3) relative to axes inclined at 60°; P is such that $PA^2-PB^2=18$; show that the locus of P is the line

$$2x+3y-12=0,$$

determine the gradient of the line and the intercepts which it makes on the x- and y-axes.

45. Find the equation of the line of gradient $\frac{3}{2}$ and passing through the point (1, 2), the axes being inclined at the acute angle whose tangent is $\frac{12}{5}$.

46. The gradient of the line

$$2x+3y-1=0$$

is the same whether the axes are rectangular or inclined at an acute angle ω; find $\tan\omega$.

47. Show that the tangent of the angle between the lines

$$A_1x+B_1y+C_1=0, \quad A_2x+B_2y+C_2=0,$$

the axes being inclined at angle ω, is

$$\pm\frac{(A_1B_2-A_2B_1)\sin\omega}{A_1A_2+B_1B_2-(A_1B_2+A_2B_1)\cos\omega}.$$

CHAPTER V

DISTANCE OF A POINT FROM A LINE

§ **28.** *If the perpendicular form of the equation of a line is*

$$x \cos \alpha + y \sin \alpha = p,$$

the distance of the point $P(x_1, y_1)$ *from the line is*

$$\pm (p - x_1 \cos \alpha - y_1 \sin \alpha),$$

according as P is on the origin or the non-origin side of the line.

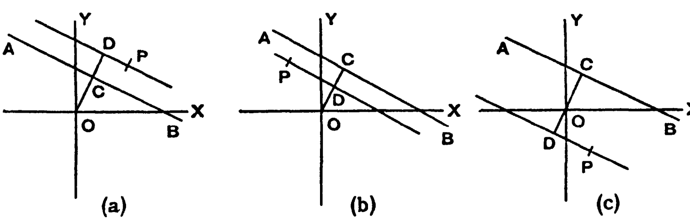

FIG. 31.

Let AB (Fig. 31) be the line whose equation in perpendicular form is

$$x \cos \alpha + y \sin \alpha = p$$

and let OC be perpendicular to AB.

Then $\quad X\hat{O}C = \alpha \quad$ and $\quad OC = p.$

The line through P parallel to AB has equation

$$x \cos \alpha + y \sin \alpha = x_1 \cos \alpha + y_1 \sin \alpha;$$

let OD be perpendicular to this line.

In case (a), where D lies on OC produced, $X\hat{O}D = \alpha$;

$$\therefore\ OD = x_1 \cos\alpha + y_1 \sin\alpha;$$

$$\therefore\ CD \text{ which} = OD - OC$$

$$= x_1 \cos\alpha + y_1 \sin\alpha - p.$$

In case (b), where D lies on OC, $X\hat{O}D = \alpha$;

$$\therefore\ OD = x_1 \cos\alpha + y_1 \sin\alpha;$$

$$\therefore\ CD \text{ which} = OC - OD$$

$$= p - x_1 \cos\alpha - y_1 \sin\alpha.$$

In case (c), where D lies on CO produced, $X\hat{O}D = \alpha + \pi$;

$$\therefore\ OD = -(x_1 \cos\alpha + y_1 \sin\alpha);$$

$$\therefore\ CD \text{ which} = OC + OD$$

$$= p - x_1 \cos\alpha - y_1 \sin\alpha.$$

$\therefore$ CD, which in all cases is equal to the perpendicular from P to AB, is

$$\pm(p - x_1 \cos\alpha - y_1 \sin\alpha),$$

according as P is on the origin or non-origin side of AB.

Note 1. Since the expression $\pm(p - x_1 \cos\alpha - y_1 \sin\alpha)$ gives the numerical length of the perpendicular from the point (x_1, y_1), it follows that the expression $p - x_1 \cos\alpha - y_1 \sin\alpha$ is positive or negative, according as the point (x_1, y_1) is on the origin or non-origin side of the line.

Note 2. The expression $p - x_1 \cos\alpha - y_1 \sin\alpha$ reduces to p when the point (x_1, y_1) is the origin; this will assist the reader to remember that the positive sign applies to points on the origin side of the line.

Note 3. It should be noted that the line divides the plane of the axes into two parts and that

$$\begin{aligned} p - x\cos\alpha - y\sin\alpha &> 0 \text{ in one part,} \\ &< 0 \text{ in other part,} \\ &= 0 \text{ on the line.} \end{aligned}$$

Note 4. If P has polar coordinates (r, θ), the distance of P from the line can be written

$$\pm\{p - r\cos(\theta - \alpha)\}.$$

§ **29.** *The distance of the point* (x_1, y_1) *from the line*

$$Ax+By+C=0$$

is

$$\pm\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

(i) C positive.—The perpendicular form of the equation of the line

$$Ax+By+C=0$$

is

$$-\frac{A}{\sqrt{A^2+B^2}}x-\frac{B}{\sqrt{A^2+B^2}}y=\frac{C}{\sqrt{A^2+B^2}};$$

$\therefore$ the distance of the point (x_1, y_1) from the line is

$$\pm\frac{C+Ax_1+By_1}{\sqrt{A^2+B^2}},$$

according as (x_1, y_1) is on the origin or non-origin side of the line.

(ii) C negative.—The perpendicular form is

$$\frac{A}{\sqrt{A^2+B^2}}x+\frac{B}{\sqrt{A^2+B^2}}y=-\frac{C}{\sqrt{A^2+B^2}}$$

and the distance is $\quad\mp\dfrac{C+Ax_1+By_1}{\sqrt{A^2+B^2}}.$

Combining these two cases, we have that the distance of the point (x_1, y_1) from the line $Ax+By+C=0$ is

$$\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}$$

where this expression and C have like signs when the point (x_1, y_1) is on the origin side of the line, and unlike signs when on the non-origin side.

Note 1. The reader is advised to neglect at first the ambiguity of sign, remembering that the numerical distance is given by the expression

$$\pm\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

Note 2. In any particular case the expression for the distance of the point (x_1, y_1) from the line $Ax+By+C=0$ can be obtained in the following manner.

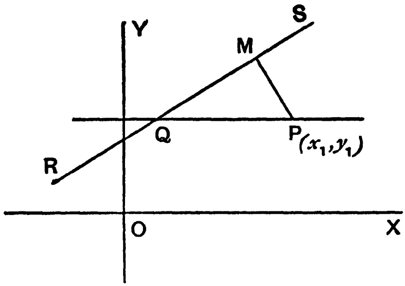

FIG. 32.

Let RS (Fig. 32) be the line, P the point and PM the perpendicular from P to RS.

Let PQ parallel to OX meet RS at Q.

Then
$$\tan P\hat{Q}M = \text{gradient of } RS$$
$$= -\frac{A}{B};$$
$$\therefore \sin P\hat{Q}M, \text{ which} = \pm\frac{\tan P\hat{Q}M}{\sqrt{1+\tan^2 P\hat{Q}M}},$$
$$= \pm\frac{A}{\sqrt{A^2+B^2}}.$$

At Q,
$$y = y_1$$
and $\therefore$
$$Ax+By_1+C=0$$
$$\therefore x = -\frac{By_1+C}{A}.$$

Hence
$$QP = x_1+\frac{By_1+C}{A}$$
$$=\frac{Ax_1+By_1+C}{A}$$
$$\therefore PM, \text{ which} = QP \sin P\hat{Q}M,$$
$$= \pm\frac{Ax_1+By_1+C}{A}\cdot\frac{A}{\sqrt{A^2+B^2}}$$
$$= \pm\frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}.$$

Example 1. *Find the distance of the point* (2, 3) *from the line*

$$7x - 24y + 8 = 0\,;$$

show that the origin and the point (2, 3) *are on opposite sides of the line.*

$$\text{Distance} = \frac{7(2) - 24(3) + 8}{\sqrt{7^2 + 24^2}} \text{ numerically}$$

$$= -\tfrac{50}{25} \qquad \text{,,}$$

$$= -2 \qquad \text{,,}$$

$\therefore$ point (2, 3) is distant 2 units from the line $7x - 24y + 8 = 0$, and as $\dfrac{7(2) - 24(3) + 8}{\sqrt{7^2 + 24^2}}$ and 8 have opposite signs the point (2, 3) is on the non-origin side of the line.

Example 2. *Show that the points* $A(3, 2)$, $B(-3, -1)$ *lie within adjacent angles formed by the lines*

$$x - 2y + 2 = 0 \quad \text{..........................(i)}$$

$$x + y + 1 = 0 \quad \text{..........................(ii)}$$

At A, $x - 2y + 2$ is +ive, $x + y + 1$ is +ive;

at B, $x - 2y + 2$ is +ive, $x + y + 1$ is −ive.

$\therefore$ A, B are on the same side of (i) and on opposite sides of (ii), i.e. A, B lie within adjacent angles formed by (i) and (ii).

§ **30.** *Bisectors of the angles between the lines*

$$Ax + By + C = 0 \quad \text{..........................(i)}$$

$$A_1x + B_1y + C_1 = 0 \quad \text{..........................(ii)}$$

Let $P(x_1, y_1)$ be any point on one of the bisectors; then the distances of P from (i) and (ii) are numerically equal.

$$\therefore \qquad \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} = \pm \frac{A_1x_1 + B_1y_1 + C_1}{\sqrt{A_1^2 + B_1^2}}$$

but this is the condition that P lies on one of the lines

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}} = \pm \frac{A_1x + B_1y + C_1}{\sqrt{A_1^2 + B_1^2}} \quad \text{...............(iii)}$$

which are $\therefore$ the required bisectors.

Note 1. In the same way the bisectors of the angles between the lines $x \cos \alpha + y \sin \alpha = p, \quad x \cos \alpha_1 + y \sin \alpha_1 = p_1$

have equations

$$p - x \cos \alpha - y \sin \alpha = \pm(p_1 - x \cos \alpha_1 - y \sin \alpha_1).$$

Note 2. According as C, C_1 have like or unlike signs, the positive or negative sign in equation (iii) gives the bisector of that angle between (i) and (ii) which contains the origin.

Example. The sides of a triangle have equations

$$x + y + 1 = 0 \quad \text{.........................(i)}$$
$$x - y + 3 = 0 \quad \text{.........................(ii)}$$
$$7x - y + 3 = 0 \quad \text{.........................(iii)}$$

find the centre of the circle escribed to (iii).

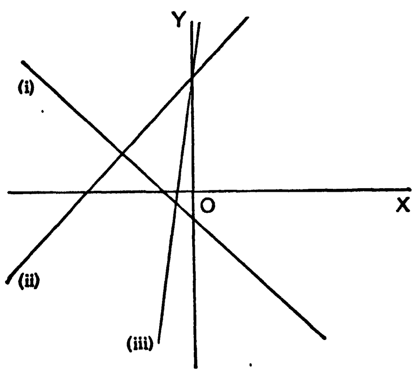

FIG. 33.

Triangle is as indicated in Fig. 33.

∴ required ex-centre is the point of intersection of the bisectors of those angles between (i) and (iii) and between (ii) and (iii), which contain the origin.

The bisector of that angle between (i) and (iii) which contains the origin is

$$\frac{x + y + 1}{\sqrt{2}} = \frac{7x - y + 3}{5\sqrt{2}}$$

i.e. $2x - 6y - 2 = 0$

i.e. $x - 3y - 1 = 0.$ ....(iv)

The bisector of that angle between (ii) and (iii) which contains the origin is

$$\frac{x-y+3}{\sqrt{2}}=\frac{7x-y+3}{5\sqrt{2}}$$

i.e. $$2x+4y-12=0$$

i.e. $$x+2y-6=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(v)$$

Where (iv) and (v) meet, $x=\dfrac{-2-18}{-3-2}=4,$

$$y=\frac{-6+1}{-3-2}=1.$$

$\therefore$ the required ex-centre is the point (4, 1).

Exercises

1. Find the distance between the point and the line in the following cases,

(i) $(2, 1)$, $3x+4y+5=0$

(ii) $(1, -2)$, $5x+12y-7=0$

(iii) $(-2, -3)$, $x-2y+6=0$

(iv) $(0. 0)$, $y=mx+a\sqrt{1+m^2}$

(v) (h, k), $y-k=m(x-h)+a\sqrt{1+m^2}$

(vi) $(-a\sin\theta, b\cos\theta)$, $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1.$

2. Find the distance of the point $\left(3, \dfrac{2\pi}{3}\right)$ from the line whose polar equation is

$$r\cos\left(\theta-\frac{\pi}{3}\right)=1.$$

3. Show that the circle with centre the origin and radius 2 touches the lines

$$3x+4y-10=0, \quad 5x-12y-26=0, \quad 4x+3y+10=0.$$

4. Show that the point $(-1, 2)$ is the incentre of the triangle with vertices $(2, 3)$, $(-2, -5)$, $(-4, 6)$.

5. The line

$$3x+4y+18=0$$

touches a circle with centre (4, 5); find the radius.

6. If p is the length of the perpendicular from the origin to the line

$$\frac{x}{a}+\frac{y}{b}=1,$$

show that
$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

7. Find the distance between the lines

$$5x-12y-5=0, \quad 5x-12y+21=0.$$

8. Show that the point $(-2, 6)$ lies on the same side of the line

$$3x+2y-7=0$$

as the origin.

9. Show that the points $(3, 2)$, $(7, 3)$ lie on opposite sides of the line

$$2x-5y+3=0.$$

10. Show that the point $(3, 2)$ lies outside the parallel lines

$$x-2y+3=0, \quad x-2y+4=0.$$

11. Show that the origin and the point $(1, 6)$ lie within vertically opposite angles between the lines

$$x-y+4=0, \quad x+2y-4=0.$$

12. Find the equations of the bisectors of the angles between the lines

$$\text{(i) } x+y-1=0, \quad x-y+1=0,$$
$$\text{(ii) } 3x-4y+2=0, \quad 4x-3y+3=0,$$
$$\text{(iii) } 4x-4y+3=0, \quad x+7y-2=0.$$

13. Show that the point $(-\frac{1}{2}, -2)$ is equidistant from the lines

$$2x-3y+4=0, \quad 6x+4y-7=0.$$

14. Show that the point $(0, 1)$ lies on the bisector of one of the angles between the lines

$$3x-4y+1=0, \quad 4x+3y-6=0.$$

15. The point (x, y) is equidistant from the lines

$$3x-4y+1=0, \quad 4x+3y+1=0\,;$$

show that

$$x+7y=0 \quad \text{or} \quad 7x-y+2=0.$$

16. Show that the lines

$$4x+3y-3=0, \quad 12x+5y-13=0$$

are tangents to the circle with centre $(-2, -3)$ and radius 4.

17. Find the equation of the line through the point (12, 7) and perpendicular to the line
$$5x+12y+25=0\,;$$
determine the point of intersection of the lines, and hence find the distance of the given point from the given line.

18. Find the equation of the line parallel to the line
$$3x+4y+4=0$$
and distant 5 units from it on the origin side.

19. The perpendiculars from P to the lines
$$x-2y+2=0, \quad x+2y+4=0$$
are numerically in the ratio 3 : 1 ; show that the locus of P is the lines
$$x+4y+5=0, \quad 2x+2y+7=0.$$

20. The perpendiculars from P to the lines
$$2x+y-1=0, \quad x+2y+1=0$$
are numerically in the ratio 2 : 1 ; show that the locus of P consists of two straight lines and find their equations.

21. Show that the locus of a point the sum of the squares of whose distances from the lines
$$x+2y-1=0, \quad 2x-y+3=0$$
is 2 has equation
$$x^2+y^2+2x-2y=0.$$

22. Show that the locus of a point which is equidistant from the point $(-1, 2)$ and the line
$$2x-y+1=0$$
has equation
$$(x+2y)^2+6(x-3y+4)=0.$$

23. A variable point P is such that its distance from the point $(2, -1)$ is equal to its distance from the line
$$3x+4y-5=0\,;$$
show that the distance of P from the line
$$7x-9y-10=0$$
varies as the square of its distance from the line
$$4x-3y=0.$$

24. A point is such that the square of its distance from the line
$$x-2y-1=0$$
exceeds by 3 the square of its distance from the line
$$2x-y+1=0\,;$$
show that the locus of the point has equation
$$x^2-y^2+2x-2y+5=0.$$

25. A variable point is such that the square of its distance from the line

$$2x + y - 1 = 0$$

exceeds by 3 the square of its distance from the line

$$x + 2y + 1 = 0\,;$$

show that the product of its distances from the lines

$$x + y + 2 = 0, \quad x - y - 3 = 0$$

varies as its distance from the line

$$x - 3y - 1 = 0.$$

CHAPTER VI

CONCURRENCE

§ **31.** *Lines concurrent with two given lines.*

Let the given lines be

$$Ax+By+C=0 \quad \text{.........................(i)}$$

and $$A_1x+B_1y+C_1=0 \quad \text{.........................(ii)}$$

Let k be any constant, and consider the equation

$$Ax+By+C+k(A_1x+B_1y+C_1)=0 \quad \text{...........(iii)}$$

which is of the first degree in x and y and therefore represents a straight line.

If (x_1, y_1) is the point of intersection of (i) and (ii)

$$Ax_1+By_1+C=0$$

and $$A_1x_1+B_1y_1+C_1=0$$

$$\therefore \; Ax_1+By_1+C+k(A_1x_1+B_1y_1+C_1)=0$$

i.e. (x_1, y_1) is a point on line (iii)

$\therefore$ equation (iii) represents for any value of k a straight line passing through the point of intersection of (i) and (ii).

Note. For different values of k, equation (iii) represents different straight lines.

Example 1. *Find the equation of the line which has gradient* $\frac{1}{2}$ *and passes through the point of intersection of the lines*

$$3x-2y=8, \quad 2x-5y=3.$$

Any line through the point of intersection of the given lines has equation of the form

$$3x-2y-8+k(2x-5y-3)=0$$

This equation represents a line of gradient $\frac{1}{2}$ if

$$\frac{3+2k}{2+5k}=\frac{1}{2}$$

i.e. if $$k=4$$

$\therefore$ required equation is

$$3x-2y-8+4(2x-5y-3)=0$$

i.e. $$11x-22y-20=0.$$

Example 2. *Find the equation of the line passing through the point* (1, 2) *and concurrent with the lines*

$$3x-4y=1, \quad 2x+3y=5.$$

Required equation is

$$3x-4y-1+k(2x+3y-5)=0$$

where $$3-8-1+k(2+6-5)=0$$

i.e. where $$k=2$$

$\therefore$ equation is

$$3x-4y-1+2(2x+3y-5)=0$$

i.e. $$7x+2y-11=0.$$

§ 32. *Consider the lines*

$$A_1x+B_1y+C_1=0 \quad \text{.......................(i)}$$
$$A_2x+B_2y+C_2=0 \quad \text{.......................(ii)}$$
$$A_3x+B_3y+C_3=0 \quad \text{.......................(iii)}$$

Lines (ii) and (iii) meet at the point whose coordinates are

$$x=\frac{B_2C_3-B_3C_2}{A_2B_3-A_3B_2}, \quad y=\frac{C_2A_3-C_3A_2}{A_2B_3-A_3B_2};$$

and this point lies on line (i) if

$$A_1\frac{B_2C_3-B_3C_2}{A_2B_3-A_3B_2}+B_1\frac{C_2A_3-C_3A_2}{A_2B_3-A_3B_2}+C_1=0$$

$\therefore$ condition that lines (i), (ii), (iii) are concurrent is

$$A_1(B_2C_3-B_3C_2)+B_1(C_2A_3-C_3A_2)+C_1(A_2B_3-A_3B_2)=0.$$

Note 1. The condition for concurrence may be written

$$A_1(B_2C_3-B_3C_2)+A_2(B_3C_1-B_1C_3)+A_3(B_1C_2-B_2C_1)=0.$$

§ **33.** *Consider the lines*

$$A_1x + B_1y + C_1 = 0 \quad \text{.......................(i)}$$
$$A_2x + B_2y + C_2 = 0 \quad \text{.......................(ii)}$$
$$A_3x + B_3y + C_3 = 0 \quad \text{.....................(iii)}$$

We shall show that these lines are concurrent if there exist three constants k, l, m (not zero), such that

$$k(A_1x + B_1y + C_1) + l(A_2x + B_2y + C_2)$$
$$+ m(A_3x + B_3y + C_3) = 0, \text{ identically} \quad \text{...........(iv)}$$

Let lines (i) and (ii) intersect at the point (x_1, y_1);

then $\qquad A_1x_1 + B_1y_1 + C_1 = 0, \quad A_2x_1 + B_2y_1 + C_2 = 0$

$\therefore$ by equation (iv) $A_3x_1 + B_3y_1 + C_3 = 0$

$\therefore$ the point (x_1, y_1) lies on line (iii)

i.e. lines (i), (ii), (iii) are concurrent.

Note. If in proving lines concurrent suitable constants k, l, m are not readily found, use the method of § 32.

Example 1. *Show that the lines*

$$x - 4y + 2 = 0, \quad 4x - y + 3 = 0, \quad x + 2y = 0$$

are concurrent.

$$3(x - 4y + 2) - 2(4x - y + 3) \equiv -5x - 10y$$
$$\equiv -5(x + 2y)$$

$\therefore$ $3(x - 4y + 2) - 2(4x - y + 3) + 5(x + 2y) \equiv 0$

$\therefore$ the given lines are concurrent.

Example 2. *Show that the lines*

$$2x + y - 5 = 0, \quad 3x - 2y - 4 = 0, \quad 4x - 9y + 1 = 0$$

are concurrent.

Where the first two meet

$$x = \frac{-4 - 10}{-4 - 3} = 2, \quad y = \frac{-15 + 8}{-4 - 3} = 1$$

When $x = 2$, $y = 1$,

$$4x - 9y + 1 = 8 - 9 + 1 = 0$$

$\therefore$ the given lines are concurrent.

Exercises

1. Find the equation of the line passing through the origin and concurrent with the lines

$$3x - 4y + 2 = 0, \quad 2x - 5y - 3 = 0.$$

2. Find the equation of the line, of gradient $\frac{4}{3}$, concurrent with the lines

$$x + y - 5 = 0, \quad x - 2y + 4 = 0.$$

3. Show that the line joining the point $(-2, -1)$ to the point of intersection of the lines

$$2x - 3y = 3, \quad x + 3y = 15$$

passes through the origin.

4. Find the equation of the line parallel to the x-axis and concurrent with the lines

$$4x + 3y - 6 = 0, \quad x - 2y - 7 = 0.$$

5. Find the equations of the lines which make numerically equal intercepts on the coordinate axes, and which are concurrent with the lines

$$2x + 3y - 1 = 0, \quad x - 2y + 3 = 0.$$

6. Find the coordinates of the foot of the perpendicular from the point of intersection of the lines

$$x + 2y = 6, \quad y - 2x = 8$$

to the line

$$3x + 4y + 15 = 0.$$

7. Show that for all values of k the equation

$$x - 3y + 9 + k(3x - 2y - 1) = 0$$

represents a line passing through a fixed point, and find the coordinates of the point.

8. Find the point of intersection of the lines

$$3x + y + 1 - a(2x - y + 4) = 0$$

and

$$b(3x + y + 1) + c(2x - y + 4) = 0.$$

9. Show that the lines

$$\begin{aligned} 2x - 3y + 1 + a(3x + y - 1) &= 0 \\ b(2x - 3y + 1) - (3x + y - 1) &= 0 \\ x - 7y + 3 &= 0 \end{aligned}$$

are concurrent.

10. Interpret the equation

$$ax + by + c - a(x - g + c) = 0.$$

11. Find the equation of the line concurrent with the lines

$$3x-y+9=0, \quad x+2y-4=0,$$

and also with the lines

$$2x+y-4=0, \quad x-2y+3=0.$$

12. Find the equations of the lines which are concurrent with the lines

$$2x-3y+1=0, \quad x+2y-3=0,$$

and which respectively have gradient 3 and pass through the point $(2, -1)$; find also the angle between these lines.

13. Show that the line passing through the point $(2, -3)$ and concurrent with the lines

$$3x+2y-2=0, \quad 4x+3y-7=0$$

has gradient $-\frac{8}{5}$.

14. Show that the line joining the points $(4, 3)$, $(-4, -1)$ is concurrent with the lines

$$2x-3y+2=0, \quad 3x-4y+2=0.$$

15. Show that the line through the point $(2, 3)$, and concurrent with the lines

$$3x+4y-7=0, \quad 2x-3y+1=0$$

is concurrent with the lines

$$3x-y-7=0, \quad 3x-2y+4=0.$$

16. BC, CA, AB are the lines

$$3x+y-21=0, \quad 2x+3y+7=0, \quad x-2y+7=0;$$

AP passes through the point $(5, -4)$; BP has gradient 2; find the equations of AP, BP, CP.

17. Prove the following lines concurrent:

(i) $4x-3y+2=0, \quad 3x-y-6=0, \quad x-y+2=0.$

(ii) $5x+2y-8=0, \quad x-5y+2=0, \quad x+7y-6=0.$

(iii) $3x-2y+3=0, \quad 5x+6y-2=0, \quad 3x+2y=0.$

(iv) $x-6y+2=0, \quad x+2=0, \quad 3x+5y+6=0.$

(v) $(p+1)x+(p-1)y+p=0, \quad (q-1)x+(q+1)y+q=0, \quad x=y.$

(vi) $(\alpha+\beta)x+\alpha\beta y+(\alpha-\beta)=0, \quad \left(\frac{1}{\alpha}-\frac{1}{\beta}\right)x+y+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)=0, \quad \alpha x=\beta.$

18. Show that the lines

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,$$

$$-\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta=1,$$

$$bx+ay\tan\left(\theta-\frac{\pi}{4}\right)=0$$

are concurrent.

19. Show that the lines

$$y=\frac{x}{t_1}+at_1, \quad y=\frac{x}{t_2}+at_2, \quad (t_1+t_2)x+(1-t_1t_2)y=a(t_1+t_2)$$

are concurrent.

20. The lines

$$x-3y+2=0, \quad x-6y+3=0, \quad x+ay=0$$

are concurrent; find the value of a.

21. The lines

$$2x+3y-5=0, \quad 3x+y+3=0, \quad x+2y-a=0$$

are concurrent; find the value of a.

22. The lines

$$3x+5y-2=0, \quad 2x+3y=0, \quad ax+by+1=0$$

are concurrent; find the relation between a and b.

23. The lines

$$ax+2y+1=0, \quad bx+3y+1=0, \quad cx+4y+1=0$$

are concurrent; show that a, b, c are in arithmetic progression.

24. The lines

$$x-8y=0, \quad x-6y-2=0, \quad x-5y-3=0, \quad x-2y-6=0$$

meet the x-axis in the origin, A_1, A_2, A_3 and the y-axis in the origin, B_1, B_2, B_3 respectively; show that (i) the lines are concurrent, (ii) $\frac{OA_1}{A_1A_2}=\frac{OA_3}{A_2A_3}$, (iii) $\frac{OB_1}{B_1B_2}=\frac{OB_3}{B_2B_3}$.

CHAPTER VII

PAIRS OF STRAIGHT LINES

§ **34.** *The equation*

$$(Ax+By+C)(A_1x+B_1y+C_1)=0 \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

represents the two straight lines

$$Ax+By+C=0 \quad \ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

and $$A_1x+B_1y+C_1=0 \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Let $P(x_1, y_1)$ be any point whose coordinates satisfy equation (i); then

$$(Ax_1+By_1+C)(A_1x_1+B_1y_1+C_1)=0$$

$\therefore$ either $Ax_1+By_1+C=0$ or $A_1x_1+B_1y_1+C_1=0$

i.e. P lies on one of the lines (ii), (iii)

$\therefore$ equation (i) represents the two lines (ii), (iii)

Example. *Find the lines represented by the equations*

(i) $x^2-y^2=0$

(ii) $x^2-7x+12=0$

(iii) $xy+2x-3y-6=0$

(iv) $x^2+3xy+2y^2+3x+5y+2=0$

Equation (i) may be written in the form

$$(x+y)(x-y)=0$$

and $\therefore$ represents the lines

$$x+y=0, \quad x-y=0.$$

Similarly equation (ii) represents the lines

$$x-3=0, \quad x-4=0,$$

equation (iii) represents the lines

$$x-3=0, \quad y+2=0,$$

and equation (iv) represents the lines

$$x+y+2=0, \quad x+2y+1=0.$$

§ **35.** *The condition that the equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{..............(i)}$$

should represent two straight lines.

Equation (i) represents two straight lines if it can be written in the form

$$(Ax+By+C)(A_1x+B_1y+C_1)=0,$$

i.e. if it can be resolved into two equations of the form

$$x=-\frac{B}{A}y-\frac{C}{A}, \quad x=-\frac{B_1}{A_1}y-\frac{C_1}{A_1} \quad \text{..............(ii)}$$

Now equation (i) can be written

$$ax^2+2(hy+g)x+(by^2+2fy+c)=0$$

i.e.
$$x=\frac{-(hy+g)\pm\sqrt{(hy+g)^2-a(by^2+2fy+c)}}{a}$$

$\therefore$ equation (i) can be written in the form (ii) if

$$(hy+g)^2-a(by^2+2fy+c)$$

i.e. $(h^2-ab)y^2+2(hg-af)y+(g^2-ac)$ is a perfect square

$\therefore$ if $\qquad (hg-af)^2-(h^2-ab)(g^2-ac)=0$

i.e. if $\qquad a(abc+2fgh-af^2-bg^2-ch^2)=0$

If $a\neq 0$, this condition reduces to

$$abc+2fgh-af^2-bg^2-ch^2=0 \quad \text{..............(iii)}$$

If $a=0$, $b\neq 0$, equation (i) can be solved as a quadratic in y and condition (iii) obtained.

If $a=0$, $b=0$, $h\neq 0$ equation (i) is

$$2hxy+2gx+2fy+c=0$$

i.e.
$$xy+\frac{g}{h}x+\frac{f}{h}y+\frac{c}{2h}=0$$

which, if factorisable, is

$$\left(x+\frac{f}{h}\right)\left(y+\frac{g}{h}\right)=0$$

and hence $$\frac{fg}{h^2}=\frac{c}{2h}$$

i.e. $$2fgh-ch^2=0$$

to which condition (iii) reduces when $a=b=0$.

$\therefore$ in all cases, the condition that equation (i) should represent two straight lines is

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Note 1. Equation (i), called the general equation of the second degree, is the most general form of equation involving terms in x and y of no higher degree than the second.

Note 2. We cannot have $a=b=h=0$, for equation (i) is then of the first degree, and represents one straight line.

Note 3. The factors of the first member of an equation of form (i) may be found either by inspection or by solving the equation as a quadratic in x or y.

Example 1. *Find the equations of the lines represented by the equation*

$$3x^2+4xy-4y^2-8x+8y-3=0 \quad \text{..............(i)}$$

Equation (i) is $(3x-2y+1)(x+2y-3)=0$

$\therefore$ represents the lines

$$3x-2y+1=0, \quad x+2y-3=0.$$

Example 2. *The equation*

$$2x^2+5xy+ky^2-5x+13y-12=0$$

represents two straight lines ; find the value of k.

$$abc+2fgh-af^2-bg^2-ch^2=0$$

where $a=2$, $f=\frac{13}{2}$

$b=k$, $g=-\frac{5}{2}$

$c=-12$, $h=\frac{5}{2}$

$\therefore$ $$-24k-\tfrac{325}{4}-\tfrac{169}{2}-\tfrac{25}{4}k+75=0,$$

i.e. $$121k=-363$$

i.e. $$k=-3.$$

§ 36. When $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv k(Ax + By + C)(A_1x + B_1y + C_1)$$

where k is a constant, we have

$$ax^2 + 2hxy + by^2 \equiv k(Ax + By)(A_1x + B_1y)$$

$\therefore$ when the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{...............(i)}$$

represents the lines

$$Ax + By + C = 0, \quad A_1x + B_1y + C_1 = 0 \quad \text{...........(ii)}$$

the equation

$$ax^2 + 2hxy + by^2 = 0 \quad \text{.....................(iii)}$$

represents the lines

$$Ax + By = 0, \quad A_1x + B_1y = 0 \quad \text{.................(iv)}$$

But lines (iv) are the parallels through the origin to lines (ii)

$\therefore$ when equation (i) represents two straight lines, equation (iii) represents the parallel lines through the origin.

Note. The equation $ax^2 + 2hxy + by^2 = 0$ may be said to represent two straight lines through the origin, for all values of a, b, h, for the equation can be written

$$by^2 + 2hxy + ax^2 = 0,$$

i.e.

$$y = \left(\frac{-h \pm \sqrt{h^2 - ab}}{b}\right)x \quad \text{..............(i)}$$

According as $h^2 > ab$, $h^2 = ab$, $h^2 < ab$, equation (i) represents two real, two coincident, or two imaginary lines through the origin; in the last case the origin is the only real point whose coordinates satisfy equation (i).

§ 37. *The angle between the lines*

$$ax^2 + 2hxy + by^2 = 0 \quad \text{.......................(i)}$$

Let equation (i) represent the lines $y = m_1x$, $y = m_2x$, and let θ be the angle between these lines;

then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_2}$$

but since equation (i) may be written

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$$

m_1, m_2 are roots of the equation

$$bm^2+2hm+a=0$$

$$\therefore \qquad m_1+m_2=-\frac{2h}{b}, \quad m_1m_2=\frac{a}{b}$$

$$\therefore \qquad \tan\theta=\pm\frac{\sqrt{\dfrac{4h^2}{b^2}-\dfrac{4a}{b}}}{1+\dfrac{a}{b}}$$

$$=\pm\frac{2\sqrt{h^2-ab}}{a+b}$$

Note 1. If $a+b=0$, $\tan\theta=\infty$ and the lines are perpendicular; if $h^2=ab$, $\tan\theta=0$ and the lines are coincident.

Note 2. It follows from § 36 that the formula for $\tan\theta$ is applicable to the general equation of the second degree when it represents two straight lines. The reader should observe, however, that in the case of the general equation representing two straight lines the relation $h^2=ab$ holds when the lines are either coincident or parallel.

Example. Given that the equation

$$2x^2-8xy+4y^2+4x+12y-23=0$$

represents two straight lines, find the tangent of the angle between them.

$$\text{Tangent of angle}=\pm\frac{2\sqrt{16-8}}{2+4}$$
$$=\pm\tfrac{2}{3}\sqrt{2}$$

The negative sign refers to the obtuse angle between the lines.

§ **38.** *The equation of the bisectors of the angles between the lines*

$$ax^2+2hxy+by^2=0 \quad \text{..............} \text{........(i)}$$

Let equation (i) represent the lines $y-m_1x=0$, $y-m_2x=0$; then the equations of the bisectors are

$$\frac{y-m_1x}{\sqrt{1+m_1^2}}=\pm\frac{y-m_2x}{\sqrt{1+m_2^2}}$$

$\therefore$ the equation of the bisectors is

$$\frac{(y-m_1x)^2}{1+m_1^2} - \frac{(y-m_2x)^2}{1+m_2^2} = 0$$

i.e. $$(1+m_2^2)(y-m_1x)^2 - (1+m_1^2)(y-m_2x)^2 = 0$$

i.e. $$(m_1^2 - m_2^2)x^2 - 2(m_1\overline{1+m_2^2} - m_2\overline{1+m_1^2})xy - (m_1^2 - m_2^2)y^2 = 0$$

i.e. $$(m_1+m_2)(x^2-y^2) = 2(1-m_1m_2)xy;$$

but m_1, m_2 are roots of the equation

$$bm^2 + 2hm + a = 0$$

$\therefore$ $$m_1 + m_2 = -\frac{2h}{b}, \quad m_1m_2 = \frac{a}{b}$$

$\therefore$ the equation of the bisectors is

$$-\frac{2h}{b}(x^2-y^2) = 2\left(1-\frac{a}{b}\right)xy$$

i.e. $$h(x^2-y^2) = (a-b)xy$$

i.e. $$\frac{x^2-y^2}{a-b} = \frac{xy}{h}$$

Note. As the bisectors are mutually perpendicular, the coefficients of x^2 and y^2 in their equation are equal in magnitude but opposite in sign.

Example. Show that the bisectors of the angles between the lines

$$2x^2 - 7xy + 4y^2 = 0$$

are the bisectors of the angles between the lines

$$3x^2 + 7xy + y^2 = 0.$$

The bisectors are

$$\frac{x^2-y^2}{2-4} = \frac{xy}{-\frac{7}{2}} \quad \text{and} \quad \frac{x^2-y^2}{3-1} = \frac{xy}{\frac{7}{2}}$$

i.e. in both cases $$7x^2 - 4xy - 7y^2 = 0.$$

§ **39.** *To determine the point of intersection of the lines*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Writing equation (i) in the form

$$ax^2 + 2(hy+g)x + (by^2 + 2fy + c) = 0 \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

we see that in general there are two values of x for any given value of y; if, however, y is the ordinate of P the point of intersection of the lines (i), equation (ii) will yield two equal values of x and these are the abscissa of P.

Now, when equation (ii) has equal roots,

$$x = -\frac{hy+g}{a}$$

$\therefore$ at P, $$ax + hy + g = 0 \quad \text{.........................(iii)}$$

Similarly, considering equation (i) in the form

$$by^2 + 2(hx + f)y + (ax^2 + 2gx + c) = 0$$

we have, at P, $$hx + by + f = 0 \quad \text{.........................(iv)}$$

Equation (i) may also be written

$$(ax + hy + g)x + (hx + by + f)y + (gx + fy + c) = 0$$

and the coordinates of P satisfy this equation, $\therefore$ from equations (iii) and (iv) we have, at P,

$$gx + fy + c = 0$$

We have, therefore, three equations, viz.

$$ax + hy + g = 0, \quad hx + by + f = 0, \quad gx + fy + c = 0,$$

all of which hold at P and any two of which suffice to determine the coordinates of P.

Note 1. Before applying these equations to an equation of the second degree it is necessary to ensure that the latter equation represents two straight lines.

Note 2. For an alternative method of deriving these equations see § 41, Example 2.

Note 3. In any particular case, the coordinates of the point of intersection of a pair of lines represented by an equation of the second degree may be determined by finding the equation of each line and solving the equations simultaneously.

Example. *Find the point of intersection of the lines*

$$x^2 - 3xy - y^2 + 2x - 3y + 1 = 0.$$

At the point of intersection,

$$x - \tfrac{3}{2}y + 1 = 0$$

and
$$-\tfrac{3}{2}x - y - \tfrac{3}{2} = 0$$

i.e.
$$2x - 3y + 2 = 0$$

and
$$3x + 2y + 3 = 0$$

$\therefore$
$$x = -1, \quad y = 0.$$

§ 40. *The equation of the pair of lines joining the origin to the points of intersection of the line*

$$lx + my = 1 \qquad \text{(i)}$$

and the lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{(ii)}$$

Consider the equation

$$ax^2 + 2hxy + by^2 + 2(gx + fy)(lx + my) + c(lx + my)^2 = 0 \qquad \text{(iii)}$$

which is homogeneous and of the second degree, and therefore represents a pair of straight lines through the origin.

If (x_1, y_1), (x_2, y_2) are the points of intersection of (i) and (ii)

$$lx_1 + my_1 = 1$$

and
$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$\therefore$
$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2(gx_1 + fy_1)(lx_1 + my_1) + c(lx_1 + my_1)^2 = 0$$

i.e. (x_1, y_1) is a point on line pair (iii); similarly (x_2, y_2) lies on (iii).

$\therefore$ equation (iii) represents the pair of lines joining the origin to the points of intersection of (i) and (ii).

Example. Find the equation of the pair of lines joining the origin to the points of intersection of the line

$$x + 2y - 5 = 0$$

and the lines

$$3xy + y^2 - 6x - 12y + 20 = 0.$$

Equation is

$$3xy + y^2 - 6(x+2y)\left(\frac{x+2y}{5}\right) + 20\left(\frac{x+2y}{5}\right)^2 = 0$$

i.e.
$$3xy + y^2 - \tfrac{2}{5}(x+2y)^2 = 0$$

i.e.
$$2x^2 - 7xy + 3y^2 = 0.$$

Exercises

1. Find the equations which represent the following pairs of lines,—

(i) $x - y = 0, \quad 3x - 2y = 0$
(ii) $2x - 3y + 1 = 0, \quad x - y + 2 = 0$
(iii) $3x + 4y - 5 = 0, \quad y = 2x$
(iv) $2x - y + 1 = 0, \quad y = 1.$

2. Find the equations of the lines represented by the equations,—

(i) $2x^2 + xy - y^2 = 0$
(ii) $3x^2 - 5xy - 2y^2 + x + 5y - 2 = 0$
(iii) $xy + 3y^2 - 4x - 13y + 4 = 0$
(iv) $(x+1)^2 - y^2 = 0$
(v) $(5x - y - 1)^2 - (x - 3y + 3)^2 = 0.$

3. Show that the following equations represent line pairs and write down the equations of the parallel line pairs through the origin,—

(i) $2x^2 - 3xy - 2y^2 + 2x + 11y - 12 = 0$
(ii) $6x^2 - 5xy - 6y^2 - 5x + 14y - 4 = 0$
(iii) $6x^2 + xy - y^2 - 3x + y = 0$
(iv) $xy - 3x + y - 3 = 0.$

4. Show that the equation

$$2x^2 - 11xy + 5y^2 - x + 23y - 10 = 0$$

represents two straight lines, and find their point of intersection.

5. Show that the equation

$$x^2 - xy - 2y^2 - 3x + 9y - 4 = 0$$

represents two straight lines, and find the equation of the line from the origin to their point of intersection.

6. Prove that the line

$$4x - 7y + 1 = 0$$

is concurrent with the lines

$$2x^2 - 7xy + 3y^2 + x - 8y - 3 = 0.$$

7. Prove that the lines

$$x^2 - y^2 + 4x + 2y + 3 = 0$$

are concurrent with the lines

$$3x^2 - 4xy - 15y^2 + 16x + 22y + 5 = 0.$$

8. Show that the area enclosed by the lines

$$x^2 + 4xy + y^2 = 0$$

and the line

$$x + y = 1$$

is $\frac{1}{2}\sqrt{3}$.

9. Find the area enclosed by the lines

$$5x^2 - 21xy + 4y^2 + 22x + 26y - 48 = 0$$

and

$$4x + 3y + 5 = 0.$$

10. Show that the area enclosed by the lines

$$ax^2 + 2hxy + by^2 = 0$$

and

$$y = k$$

is

$$\frac{k^2}{a}\sqrt{h^2 - ab}.$$

11. Find the area of the parallelogram formed by the lines

$$3x^2 - 7xy + 2y^2 + 5x + 15y - 50 = 0$$

and the parallels through the origin.

12. The following equations represent line pairs; find the values of k and the equation of each line,—

(i) $kx^2 - xy - 6y^2 - x + 5y - 1 = 0$

(ii) $6x^2 - 13xy + ky^2 + x + 23y - 12 = 0$

(iii) $12x^2 + 19xy + 4y^2 - 5x - 11y + k = 0$

(iv) $x^2 + 6xy + 5y^2 + 8x + ky + 12 = 0$

(v) $3x^2 + 4xy + y^2 + kx + 6y + 8 = 0$

(vi) $2x^2 + kxy - 6y^2 + 3x + y + 1 = 0.$

13. Find the tangent of the angle between the lines,—

(i) $6x^2 - 5xy + y^2 = 0$

(ii) $3x^2 - xy - 2y^2 + 7x + 3y + 2 = 0$

(iii) $10x^2 - xy - 3y^2 - 6x + 8y - 4 = 0.$

14. Show that the angle between the lines

$$3x^2 - 2xy - y^2 = 0$$

is equal to the angle between the lines

$$8x^2 + 2xy - 3y^2 + 2x + 4y - 1 = 0.$$

15. Show that the lines

$$bx^2 - 2hxy + ay^2 = 0$$

are perpendicular to the lines

$$ax^2 + 2hxy + by^2 = 0.$$

16. Write down the equations of the line pairs which pass through the origin and which are perpendicular to the following line pairs,—

(i) $6x^2 - xy - y^2 = 0$

(ii) $2x^2 + 3xy + y^2 + 3x + y - 2 = 0$

(iii) $2x^2 - xy - 3y^2 + 4x + 4y = 0.$

17. The gradient of one of the lines

$$ax^2 + 2hxy + by^2 = 0$$

is twice that of the other ; prove that

$$8h^2 = 9ab.$$

18. Show that the condition that the gradient of one of the lines

$$ax^2 + 2hxy + by^2 = 0$$

should be k times that of the other is

$$4kh^2 = ab(1 + k)^2.$$

19. The line pairs

$$ax^2 + 2hxy + by^2 = 0$$
$$a_1x^2 + 2h_1xy + b_1y^2 = 0$$

have a line in common ; show that

$$(ab_1 - a_1b)^2 + 4(ah_1 - a_1h)(bh_1 - b_1h) = 0.$$

20. Find the equations of the bisectors of the angles between the lines,—

(i) $6x^2 + 11xy + 3y^2 = 0,$

(ii) $5x^2 + 2xy - 4y^2 = 0,$

(iii) $7x^2 + 11xy - 6y^2 = 0.$

21. If the bisectors of the angles between the lines

$$ax^2 + 2hxy + by^2 = 0$$

are the same as those between the lines

$$a_1x^2 + 2h_1xy + b_1y^2 = 0,$$

show that $$h(a_1 - b_1) = h_1(a - b).$$

22. Prove that the bisectors of the angles between the lines

$$(a - k)x^2 + 2hxy + (b - k)y^2 = 0$$

do not vary with k.

23. Show that the equation

$$(a-k)x^2+2hxy+(b-k)y^2=0$$

represents, for any value of k, a pair of lines equally inclined to the lines

$$hx^2-(a-b)xy-hy^2=0.$$

24. Show that the lines

$$ax^2+2hxy+ay^2+2gx+2fy+c=0$$

cut the x- and y-axes in concyclic points.

25. Find the perpendicular distance between the lines

$$(3x+4y)^2-21x-28y-144=0.$$

26. Find the equation of the lines joining the origin to the point of intersection of the lines

$$3x^2-5xy-2y^2+4x+13y-15=0$$

and the line

$$2x+3y-1=0.$$

27. Show that the origin, the point of intersection of the lines

$$2x^2-7xy+3y^2+5x+10y-25=0,$$

and the points at which these lines are cut by the line

$$x+2y-5=0$$

are the vertices of a parallelogram.

28. Prove that the lines joining the origin to the points at which the line

$$6x-y+8=0$$

meets the lines

$$3x^2+4xy-4y^2-11x+2y+6=0$$

are equally inclined to each of the coordinate axes.

29. Find the points of intersection of the following pairs of lines,—

(i) $2x^2-5xy+3y^2+6x-7y+4=0$

(ii) $x^2+xy-2y^2+x+14y-20=0$

(iii) $6x^2-7xy-3y^2-9x-25y-42=0$

(iv) $2x^2+2xy-5x-3y+3=0.$

30. Show that the lines

$$6x^2+xy-y^2-14x+3y+4=0$$

and

$$6x^2-19xy+10y^2+26x-21y+8=0,$$

are concurrent.

31. Find the tangent of the angle between the lines

$$6x^2-xy-y^2+7x-y+2=0$$

the axes being inclined at the acute angle whose tangent is $\frac{3}{4}$.

CHAPTER VIII

CHANGE OF AXES

§ **41.** *Translation of Axes. If P be the point* (x, y) *and* O_1 *the point* (h, k) *relative to axes* OX, OY, *and if P is the point* (x_1, y_1) *relative to axes* O_1X_1, O_1Y_1 *parallel to* OX, OY *respectively,*

$$x = x_1 + h, \quad y = y_1 + k.$$

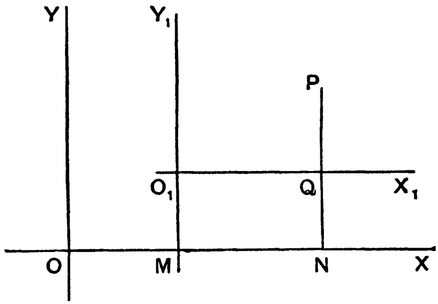

FIG. 34.

Let O_1Y_1 (Fig. 34) meet OX at M, and let the ordinate of P meet OX at N and O_1X_1 at Q; then, for all positions of O_1 and P,

$$ON = OM + MN$$

i.e. $$x = h + x_1.$$

Similarly $$y = k + y_1.$$

Note. If the equation of a locus is known and the origin is changed to the point (h, k), the new equation of the locus will be obtained by substituting $x + h$ for x and $y + k$ for y.

Example 1. *Find the equation of the line*

$$3x-y-2=0$$

referred to a new origin at the point (1, 2), *the old and new axes being parallel.*

The new equation is

$$3(x+1)-(y+2)-2=0$$

i.e. $$3x-y-1=0.$$

Example 2. *The equation*

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents two straight lines; show that the coordinates of the point of intersection satisfy the equations

$$ax+hy+g=0,\ hx+by+f=0,\ gx+fy+c=0.$$

Change the origin to the point (x_1, y_1) and the new equation of the lines is

$$a(x+x_1)^2+2h(x+x_1)(y+y_1)+b(y+y_1)^2 \\ +2g(x+x_1)+2f(y+y_1)+c=0$$

i.e. $$ax^2+2hxy+by^2+2(ax_1+hy_1+g)x+2(hx_1+by_1+f)y \\ +ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \text{......(i)}$$

If the new origin is the point of intersection of the lines, the independent term and the coefficients of x and y in equation (i) must be zero,

i.e. $$ax_1+hy_1+g=0 \quad \text{........................(ii)}$$

$$hx_1+by_1+f=0 \quad \text{........................(iii)}$$

and $$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0 \quad \text{.........(iv)}$$

Equation (iv) can be written

$$(ax_1+hy_1+g)x_1+(hx_1+by_1+f)y_1+(gx_1+fy_1+c)=0$$

$\therefore$ from (ii) and (iii)

$$gx_1+fy_1+c=0.$$

§ **42.** *Rotation of Axes. If P is the point (x, y) relative to rectangular axes OX, OY, and the point (x_1, y_1) relative to rectangular axes OX_1, OY_1 where $X\hat{O}X_1 = \theta$,*

$$x = x_1 \cos\theta - y_1 \sin\theta,$$
$$y = x_1 \sin\theta + y_1 \cos\theta.$$

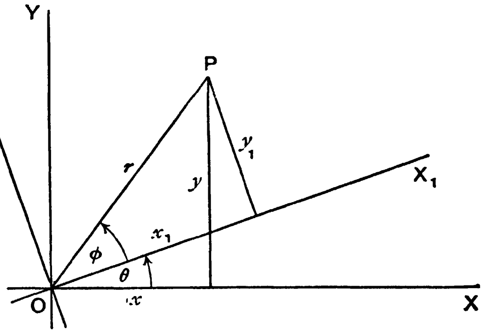

FIG. 35

Let $X_1\hat{O}P$ (Fig. 35) $= \varphi$ and let $OP = r$.

Then
$$\begin{aligned} x &= r\cos(\theta+\varphi) \\ &= r\cos\theta\cos\varphi - r\sin\theta\sin\varphi \\ &= x_1\cos\theta - y_1\sin\theta \end{aligned}$$
and
$$\begin{aligned} y &= r\sin(\theta+\varphi) \\ &= r\sin\theta\cos\varphi + r\cos\theta\sin\varphi \\ &= x_1\sin\theta + y_1\cos\theta. \end{aligned}$$

Note 1. If the equation of a locus is known and the axes are rotated through an angle θ, the new equation of the locus will be obtained by substituting $x\cos\theta - y\sin\theta$ for x and $x\sin\theta + y\cos\theta$ for y.

Note 2. From the expressions for x and y we get

$$x_1 = x\cos\theta + y\sin\theta,$$
$$y_1 = -x\sin\theta + y\cos\theta.$$

The latter expressions may also be obtained by considering

$$x_1 = r\cos(\overline{\theta+\varphi} - \theta), \quad y_1 = r\sin(\overline{\theta+\varphi} - \theta).$$

Example. Find the equation of the line pair

$$4x^2 - 11xy + 6y^2 = 0$$

when the axes are rotated through the acute angle whose tangent is $\frac{4}{3}$.

If θ is the angle, $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$.

The new equation is

$$4(\tfrac{3}{5}x - \tfrac{4}{5}y)^2 - 11(\tfrac{3}{5}x - \tfrac{4}{5}y)(\tfrac{4}{5}x + \tfrac{3}{5}y) + 6(\tfrac{4}{5}x + \tfrac{3}{5}y)^2 = 0$$

i.e. $$4(3x - 4y)^2 - 11(3x - 4y)(4x + 3y) + 6(4x + 3y)^2 = 0$$

i.e. $$125xy + 250y^2 = 0$$

i.e. $$xy + 2y^2 = 0.$$

Exercises

1. The origin is changed to the point (1, 2); find the new coordinates of the points (4, 3), (0, 4), (3, 2).

2. When the origin is changed, the point (1, 3) becomes the point (3, 4); find the coordinates of the new origin relative to the original axes.

3. When the origin is changed, the points $(-1, 3)$, $(4, -2)$ become the points $(-3, 4)$, (α, β); find α and β.

4. When the origin is changed, the points $(3, -7)$, $(-2, 5)$ become the points $(\alpha, 5)$, $(3, \beta)$; find α and β.

5. The origin is changed to the point (3, 2); find the new equations of the lines

(i) $x - 3y + 6 = 0$
(ii) $2x + 3y - 12 = 0$
(iii) $2x - 3y + 3 = 0$.

6. When the origin is changed, the line

$$x - 2y + 2 = 0$$

becomes the line

$$x - 2y - 3 = 0;$$

find the locus of the new origin relative to the original axes.

7. The origin is changed to the point (2, 1); find the new equations of the line pairs

(i) $3x^2 - 5xy + y^2 = 0$
(ii) $4x^2 - y^2 - 8x + 4 = 0$
(iii) $x^2 - 3xy + 2y^2 - x + 2y = 0$

8. The origin is changed to the point (α, β); find the new equation of the lines

$$3x^2 + xy - 2y^2 - 8x + 7y - 3 = 0$$

and hence determine the coordinates of their point of intersection relative to the original axes.

9. The origin is changed to the point $A(3, 2)$; find the new equation of the line

$$4x + 3y + 7 = 0,$$

and hence determine the length of the perpendicular from A to the given line.

10. Transform the equation

$$3x - y + 4 = 0,$$

the axes being turned through 45°.

11. Transform the equation

$$2x^2 - 3xy + y^2 = 0,$$

the axes being turned through the acute angle whose tangent is $\frac{1}{2}$.

12. Transform the equation

$$x^2 + 3xy + 2y^2 = 0,$$

the axes being turned through the obtuse angle whose tangent is $-\frac{1}{2}$.

13. Transform the equation

$$x \cos \alpha + y \sin \alpha = p,$$

the axes being turned through the angle α, and hence show that p is the length of the perpendicular from the origin to the given line.

14. Transform the equation

$$12x^2 - 7xy - 12y^2 - 32x - 24y = 0,$$

the axes being turned through the acute angle whose tangent is $\frac{3}{4}$.

15. Transform the equation

$$2x^2 - 3xy - 2y^2 + 2x + 11y - 12 = 0,$$

the origin being changed to the point (1, 2) and the axes then rotated through the acute angle whose tangent is 3.

REVISION EXERCISES

A. On Chapters I-III

1. Find the projections on the x- and y-axes of the line joining the point $(a \cos \theta, b \sin \theta)$ to the point $(-a \sin \theta, b \cos \theta)$, and show that they are equal if

$$\tan \theta = \frac{b+a}{b-a}.$$

2. A, B are the points (r_1, θ_1), (r_2, θ_2) and $M(r, \theta)$ is the mid-point of AB; show that

$$r = \tfrac{1}{2}\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

and
$$\tan \theta = \frac{r_1 \sin \theta_1 + r_2 \sin \theta_2}{r_1 \cos \theta_1 + r_2 \cos \theta_2}.$$

3. P, Q, T are the points $(a \cos \theta, b \sin \theta)$, $(-a \sin \theta, b \cos \theta)$, $\{a(\cos \theta - \sin \theta), b(\cos \theta + \sin \theta)\}$, and O is the origin; show that

$$OP^2 + OQ^2 \quad \text{and} \quad TP^2 + TQ^2$$

are equal and independent of θ.

4. Using polar coordinates, A, B, C are the points

$$\left(1, \frac{\pi}{6}\right), \left(2, \frac{5\pi}{6}\right), \left(3, \frac{7\pi}{6}\right);$$

show that $AB = BC$.

5. Show that the gradient of the line joining the points

$$(a \cos \theta, b \sin \theta), \ (-a \sin \theta, b \cos \theta)$$

is
$$-\frac{b}{a} \cot\left(\theta + \frac{\pi}{4}\right).$$

6. Show that the mid-point of the join of the points

$$\left(\frac{a \cos \theta}{1 - \sin \theta}, \frac{b \cos \theta}{1 - \sin \theta}\right) \quad \text{and} \quad \left(\frac{a \cos \theta}{1 + \sin \theta}, -\frac{b \cos \theta}{1 + \sin \theta}\right)$$

is the point $(a \sec \theta, b \tan \theta)$.

7. The vertices of a triangle are $A(-7, 3)$, $B(-4, -1)$, $C(4, 5)$; show that $\hat{B} = 90°$; hence find the circumcentre and verify that $D(\frac{7}{2}, \frac{3}{2})$ lies on the circumcircle of $\triangle ABC$.

8. A, B, C are the points $(-9, 8)$, $(-5, -4)$, $(-1, 4)$; P divides AB internally in the ratio $3:1$ and Q divides AC externally in the ratio $3:1$; find the coordinates of P and Q, and the lengths of AB, AC, AP, AQ and verify that $AB : AC = AQ : AP$.

9. A, B, C are the points $(-6, -1)$, $(0, 2)$, $(-4, -2)$; P, Q divide AB internally and externally in the ratio $2:1$, and R divides AC externally in the ratio $3:2$; find the coordinates of P, Q, R and show that $PC \parallel QR$.

10. A, B, C, D are the points $(-3, 3)$, $(12, 6)$, $(-4, -3)$, $(1, -2)$; E divides AB internally in the ratio $2:1$; F divides CD externally in the ratio $3:2$; find the coordinates of E and F, and show that $EF \parallel AD$.

11. A, B, C, D are the points $(2, -4)$, $(-3, 6)$, $(2, 4)$, $(-7, -2)$; P divides AB in the ratio $3:2$; show that P lies on CD and find the ratio $CP : PD$.

12. Show that the points $(-5, 2)$, $(4, -1)$ lie on the circle which has as diameter the line joining the points $(-4, 3)$, $(2, -5)$.

13. The vertices of a triangle are $(b, 0)$, $(a, 0)$, $\left(c, \dfrac{a+b}{2}\right)$; find its area.

14. Show that the triangles with vertices

(i) $(2, -4)$, $(4, -3)$, $(5, 5)$ (ii) $(4, -2)$, $(5, 0)$, $(1, 7)$

are equal in area.

15. Show that the triangles with vertices

(i) $(3, -5)$, $(5, -4)$, $(6, 4)$ (ii) $(4, -6)$, $(6, -2)$, $(-2, 12)$

are similar, and determine the ratio of their areas.

16. Show that the area of the triangle with vertices $(at^2, 2at)$, $(at_1^2, 2at_1)$, $\{att_1, a(t+t_1)\}$ is numerically $\dfrac{a^2}{2}(t-t_1)^3$.

17. A, B, C are the points (a, bc), (b, ca), (c, ab); show that the area of $\triangle ABC$ is $\frac{1}{2}(b-c)(c-a)(a-b)$.

18. Find the areas of the triangles the polar coordinates of whose vertices are

(i) $\left(1, \frac{\pi}{6}\right)$, $\left(3, \frac{\pi}{3}\right)$, $\left(2, \frac{\pi}{2}\right)$ (ii) $\left(1, \frac{\pi}{3}\right)$, $(2, \pi)$, $\left(4, \frac{11\pi}{6}\right)$.

19. A, B, C are the points $(-2, 3)$, $(1, -1)$, $(-2, -2)$; determine the coordinates of P such that

$$\triangle PBC = \triangle PCA = \triangle PAB$$

in magnitude and sign.

20. A, B, C are the points (3, 1), (1, 4), (0, -1);

$$6\triangle ABC = 3\triangle ABP = 2\triangle APC$$

in magnitude and sign; find the coordinates of P.

B. On Chapters IV-VI

21. A, B, C are the points (1, 2), (2, -3), (-2, 3); P moves so that

$$PA^2 + PB^2 = 2PC^2;$$

find the locus of P.

22. A variable line APB, of constant length, meets the x-axis at A and the y-axis at B; $AP = b$, $PB = a$; show that the locus of P has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

23. Write down the coordinates of P, the mid-point of the line joining the points $(2k-1, 3-k)$, $(3k-1, 2-k)$, and show that if k varies the locus of P has equation

$$4x + 10y - 21 = 0.$$

24. A, B are the points (4, -7), (3, 6); find the locus of a point P which moves so that the area of $\triangle ABP$ is numerically 10.

25. A, B, C are the points (-2, 4), (1, 1), (4, 2); find the locus of a point P which moves so that the area $PABC$ is 16.

26. AB, a straight line of constant length $2a$, meets the coordinate axes at A, B; on AB and on the side remote from the origin an equilateral triangle ABP is drawn; show that, if P is the point (x, y),

$$x^4 - x^2y^2 + y^4 - 2a^2x^2 - 2a^2y^2 + a^4 = 0.$$

27. Show that the polar equation of the line passing through the points (r_1, θ_1), (r_2, θ_2), is

$$\frac{1}{r}\sin(\theta_2 - \theta_1) + \frac{1}{r_1}\sin(\theta - \theta_2) + \frac{1}{r_2}\sin(\theta_1 - \theta) = 0.$$

28. Find the length of the perpendicular from the point $\left(4, \frac{2\pi}{3}\right)$ to the line

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 1.$$

29. Show that the point $(am^2, 2am)$ is equidistant from the point $(a, 0)$ and the line $x + a = 0$.

30. The line PQ passes through the point (-5, 2) and makes on the axes intercepts equal in magnitude and sign; find the equation of PQ and of the perpendicular from the origin to PQ.

31. A, B are the points (4, 4), (2, 5) and O is the origin;

$$\triangle OAC = 2\triangle ACB$$

in magnitude and sign; show that the locus of C is a line parallel to the y-axis.

32. The line through the point $P(a\cos\theta, 0)$ perpendicular to the x-axis meets the lines

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \quad \frac{x}{a}\cos\theta + \frac{y}{a}\sin\theta = 1$$

at Q, R respectively; show that $PR : PQ = a : b$.

33. Show that the lines

$$Ax + By + C = 0, \quad A_1x + B_1y + C_1 = 0$$

cut the axes in concyclic points if $AA_1 = BB_1$.

34. The lines

$$ax + by + c = 0, \quad a_1x + b_1y + c_1 = 0,$$
$$ax + by + c + a(a_1x + b_1y + c_1) = 0$$

meet the x-axis at A, B, C; show that $AC = a_1CB$.

35. The lines

$$ax + by + c = 0, \quad a_1x + b_1y + c_1 = 0,$$
$$ax + by + c \pm k(a_1x + b_1y + c_1) = 0$$

meet the x-axis at A, B, C, D; prove that (i) $AC : CB = ka_1 : a$; (ii) $AC : CB = AD : BD$.

36. Find the circumcentre of the triangle with vertices (3, 1), (1, 5), (− 6, 4).

37. Find the circumcentre and circumradius of the triangle with sides

$$x + 3y = 0, \quad 2x + y - 10 = 0, \quad x - 7y + 10 = 0.$$

38. A, B are the points (1, −2), (− 3, 4); find the points P on the line

$$x - 2y + 4 = 0,$$

such that the area of $\triangle PAB$ is numerically 13.

39. The line

$$\frac{x}{a} + \frac{y}{b} = 1$$

meets the x- and y-axes at A, B respectively; perpendiculars from A, B to the variable line

$$y = mx$$

meet this line at C, D; DP, CP are parallel to the x- and y-axes respectively; show that P lies on the line AB.

40. Find the equation of the line joining the points $A(-2a, 0)$, $B(0, a)$ and of the parallel line through the point $(0, 2a)$; show that, if a varies, the locus of the mid-point of AB is the line

$$x+2y=0.$$

41. A, B, C are the points $(4, 5)$, $(-3, 1{\cdot}5)$, $(3, -3)$; CD, perpendicular to AB, meets AB at D; find the coordinates of D and of E which divides CD internally in the ratio $2:1$; show also that $AE \perp BC$.

42. Show that the lines

$$4x+y-9=0,$$
$$x-2y+3=0,$$
$$5x-y-6=0$$

make equal intercepts on any line of gradient 2.

43. A variable line passes through the point $P(k+3, 2k-1)$ and has gradient $-\frac{3}{4}$; Q is a point on the line such that PQ is bisected by the y-axis; show that the locus of Q is the line

$$7x+2y+14=0.$$

44. A, B, C, D are the points $(0, 5)$, $(2, 4)$, $(2, -2)$, $(-6, -4)$; AD, BC meet at E; AB, CD meet at F; find the coordinates of E and F, and show that the mid-points of AC, BD, EF are collinear.

45. A straight line passing through the point $(1, 2)$ is terminated by the x- and y-axes; show that the locus of the mid-point of the line has equation,

$$2xy-2x-y=0.$$

46. A straight line passing through the point $(2, 3)$ meets the x-axis at A and the y-axis at B; P divides AB externally in the ratio $2:3$; show that the locus of P has equation,

$$xy+6(x-y)=0.$$

47. AD, BE, CF are the perpendiculars of the $\triangle ABC$; A, D, E, F are the points $(-4, 5)$, $(\frac{16}{5}, -\frac{23}{5})$, $(4, 1)$, $(-1, -4)$; find the equations of the sides of the triangle and the coordinates of B and C.

48. A, B are the points $(-1, 2)$, $(-3, -2)$; find the equation of (i) the line passing through A and B, (ii) the line through A and having gradient $-\frac{4}{3}$, and determine the length of the intercept which these lines make on the line,

$$2x+4y-11=0.$$

49. P, Q, two points on the line,

$$x-y+1=0,$$

are distant 5 units from the origin O; find the area of $\triangle OPQ$.

50. A line through $A(-2, -3)$ meets at B and C the lines,

$$x+3y-9=0, \quad x+y+1=0,$$

and $AB \cdot AC=20$; find the equation of AB.

51. Show that if any line through the variable point $A(k+1, 2k)$ meets the lines,

$$7x+y-16=0, \quad 5x-y-8=0, \quad x-5y+8=0$$

at B, C, D respectively, AC, AB, AD are in harmonic progression.

52. A line through $A(-5, -4)$ meets the lines,

$$x+3y+2=0, \quad 2x+y+4=0, \quad x-y-5=0$$

at B, C, D respectively; AC, AB, AD are in harmonic progression; find the equation of AB.

53. Show that the points $(at^2, 2at)$, $\{att_1, a(t+t_1)\}$ lie on the line,

$$y=\frac{x}{t}+at,$$

and show that the tangent of the angle between the lines joining these points to the point $(a, 0)$ is $\pm\dfrac{t-t_1}{1+tt_1}$.

54. PQ, RS, AB are respectively the lines,

$$ax-11y-19=0, \quad 2x-y=0, \quad 4x+3y=0;$$

PQ, RS are equally inclined to AB; find a.

55. The axes being inclined at angle ω, show that the tangent of the angle between the lines,

$$y=mx+c, \quad y=m_1x+c_1$$

is

$$\pm\frac{(m-m_1)\sin\omega}{1+(m+m_1)\cos\omega+mm_1}.$$

56. Prove that the area of the triangle whose sides are

$$a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0, \quad a_3x+b_3y+c_3=0$$

is

$$\frac{1}{2}\frac{\{a_1(b_2c_3-b_3c_2)+a_2(b_3c_1-b_1c_3)+a_3(b_1c_2-b_2c_1)\}^2}{(a_2b_3-a_3b_2)(a_3b_1-a_1b_3)(a_1b_2-a_2b_1)}.$$

57. A, C, B are the points $(-a, 0)$, $(a\cos\varphi, 0)$, $(a, 0)$; the lines

$$x\cos\varphi+y\sin\varphi=a, \quad x\cos\varphi-y\sin\varphi=a$$

meet at D; show that D lies on the x-axis and that

$$AC : CB=AD : BD.$$

58. A, B, C, D are the points

$$(4a, 4a),\ (9a, 6a),\ \left(\frac{a}{9}, -\frac{2a}{3}\right),\ \left(\frac{a}{4}, -a\right);$$

show that AB and CD meet on the line,

$$x + a = 0,$$

and that AD, BC meet at the point $(a, 0)$.

59. A, B, C, P are the points $(2, 6)$, $(-2, -2)$, $(-1, 5)$, $(-3, 1)$; D, E, F are the feet of the perpendiculars from P to BC, CA, AB; find the coordinates of D, E, F and verify that these points are collinear.

60. A, B, C are the points $(2, 5)$, $(-1, -1)$, $(-2, 3)$; P divides AB internally in the ratio $2:1$, and Q divides AC externally in the ratio $3:1$; find the coordinates of P and Q, and show that PC is parallel to BQ. A line through Q, parallel to AB, meets PC at R; show that R divides PC externally in the ratio $3:1$.

61. A, B, C are the points $(4, 6)$, $(-5, 1)$, $(2, 4)$; D divides BC in the ratio $1:2$; E divides CA externally in the ratio $3:1$; DE cuts AB at F; find the coordinates of F and show that $AF : FB = 2:3$.

62. A, B, C are the points $(-3, -2)$, $(-4, 1)$, $(-2, 5)$ and AD, BE are perpendiculars of $\triangle ABC$; find the circumcentre S of the triangle and prove that $SC \perp DE$.

63. A, B, C, D are the points $(-1, 4)$, $(-7, -2)$, $(3, -4)$, $(5, 1)$; find the area of the quadrilateral $ABCD$, and show that it is bisected by BD and also by the line,

$$5x + y + 1 = 0.$$

64. Show that the ratio of the lengths of the perpendiculars from the points $A\left(1, -\frac{1}{m}\right)$, $B\left(-\frac{1}{l}, 1\right)$ to the line,

$$lx + my + 1 = 0,$$

is $l : m$. Find the gradients of the lines passing through the origin the perpendiculars from A and B to which are in the ratio $l : m$.

65. One bisector of the angle between the lines,

$$3x - 4y + 5 = 0,\quad 5x + 12y - 1 = 0,$$

meets the x- and y-axes at A, B respectively, and the other meets these axes at A_1, B_1 respectively; show that $AA_1 : B_1B = 1 : 2$.

66. One bisector of the angle between the lines,

$$x + 8y + 18 = 0,\quad 7x + 4y + 30 = 0,$$

meets the x- and y-axes at A, B respectively, and the other meets these axes at A_1, B_1 respectively; show that AB_1, A_1B meet at right angles at the point $\left(-\frac{14}{5}, \frac{8}{5}\right)$.

67. Express in perpendicular form the equations of the bisectors of the angles between the lines,

$$x \cos \alpha + y \sin \alpha = p, \quad x \cos \alpha_1 + y \sin \alpha_1 = p_1.$$

68. AB, BC, CA are the lines,

$$x + 2y - 3 = 0, \quad 2x + y + 1 = 0, \quad x + y + 2 = 0$$

respectively ; prove that $AB = BC$.

69. Find the bisectors of the angles between the lines,

$$x - y + 2 = 0, \quad 7x - y - 16 = 0,$$

and show that these bisectors make equal intercepts on the lines,

$$3x - 4y + 1 = 0, \quad 4x + 3y - 17 = 0.$$

70. Find the coordinates of the points at unit distance from the lines,

$$3x - 4y + 1 = 0, \quad 8x + 6y + 1 = 0.$$

71. The line,

$$x - 3y + 1 = 0,$$

bisects the angle between a line L and the line,

$$x - y + 1 = 0 \; ;$$

find the equation of L.

72. Find the equation of the line joining the origin to the point of intersection of the lines,

$$x + 4y - 2a = 0, \quad x - 4y + 2a = 0.$$

73. Show that the variable line,

$$(k - 1)x + (k + 1)y - 2k = 0,$$

where k may have any value, passes through a fixed point, and find the point.

74. Determine the fixed point through which the line,

$$(2k - 3)x + (3k - 2)y - (4k - 1) = 0,$$

passes whatever the value of k may be.

75. Show that the lines,

$$\begin{aligned} 7x + 4y - 21 &= 0 \\ 2x + y - 5 &= 0 \\ 7x + y &= 0 \end{aligned}$$

are concurrent.

76. The lines,

$$\begin{aligned} 4x - 5y - 1 &= 0 \\ 5x - 8y + a &= 0 \\ x - y &= 0 \end{aligned}$$

are concurrent ; find a.

77. Show that the lines,

$$3x-4y+3=0$$
$$5x-5y+4=0$$
$$x+2y-1=0$$

are concurrent.

78. Show that the lines,

$$(b+c)x+(c+a)y-1=0$$
$$(b-c)x+(c-a)y+1=0$$
$$bx+cy=0$$

are concurrent.

79. Show that the lines,

$$(p+q)x+(p+q)y-(p-q)=0$$
$$(p-q)x-(p-q)y-(p+q)=0$$
$$px+qy-p=0$$
$$qx+py+q=0$$

are concurrent.

80. The lines,

$$ax+hy+g=0$$
$$hx+by+f=0$$
$$gx+fy+c=0$$

are concurrent; show that

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

C. On Chapters VII-VIII

81. The sides of a triangle are the lines

$$y=5 \quad \text{and} \quad 20x^2-xy-12y^2=0;$$

find the coordinates of its orthocentre.

82. The line PQ makes intercepts 2 and 3 on the x- and y-axes respectively, and meets the line pair

$$9x^2-3xy-2y^2=0$$

in P and Q; find the area of $\triangle OPQ$ where O is the origin.

83. The equation

$$ax^2+3xy-2y^2-5x+5y+c=0$$

represents two straight lines perpendicular to each other; find a and c.

84. The equation

$$3x^2+2hxy-28y^2+6x-17y+c=0$$

represents two straight lines which intersect on the x-axis; find h, c and the angle between the lines.

85. Show that the line

$$x+2y-7=0$$

is concurrent with the lines

$$3x^2-8xy+4y^2+8x-4y-3=0.$$

86. Show that each of the equations

$$3x^2+2xy-y^2+10x+6y+7=0$$

and $$2x^2+7xy-15y^2+x+44y-21=0$$

represents a pair of straight lines; prove that the four lines are concurrent.

87. Show that the line joining the origin to the point of intersection of the line pair

$$3x^2+xy-2y^2-8x+7y-3=0$$

passes through the point of intersection of the line pair

$$8x^2-6xy-2y^2+2x-7y-3=0.$$

88 Show that the lines

$$ax^2+2hxy+ay^2+2gx+2fy+c=0$$

cut the x- and y-axes at points which lie on the curve

$$ax^2+ay^2+2gx+2fy+c=0.$$

89. Find the ratios in which the lines

$$2x^2-7xy+3y^2+6x+2y-8=0$$

divide the line joining the points $(-3, 2)$, $(3, -1)$.

90. P is the point of intersection of the line pair

$$x^2+2xy-3y^2-4x+3=0;$$

one line meets the x- and y-axes in A and B, the other meets the x- and y-axes in A_1 and B_1 respectively; if A_1B and AB_1 meet in Q, show that PQ is perpendicularly bisected by the x-axis.

91. Of the line pair

$$2x^2+xy-3y^2-10x+12=0,$$

one line meets the x- and y-axes in A and B, the other meets the x- and y-axes in A_1 and B_1 respectively; show that AB_1 and A_1B meet at the same angle as the given lines.

92. The line pairs

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and $$ax^2+2hxy+by^2+2g_1x+2f_1y+c_1=0$$

have a line in common; show that the line has equation

$$2(g-g_1)x+2(f-f_1)y+(c-c_1)=0.$$

93. Show that the line

$$2(g-g_1)x+2(f-f_1)y+(c-c_1)=0$$

is a diagonal of the parallelogram formed by the lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and $$ax^2+2hxy+by^2+2g_1x+2f_1y+c_1=0.$$

94. Given that the equation

$$(ax+by)^2+2gx+2fy+c=0$$

represents two straight lines, show that

(i) the lines are parallel

(ii) $af=bg$

(iii) the distance between the lines is $\dfrac{2}{a}\sqrt{\dfrac{g^2-a^2c}{a^2+b^2}}$.

95. Show that the equation

$$(\beta^2-1)x^2-2\alpha\beta xy+(\alpha^2-1)y^2+2\alpha x+2\beta y-(\alpha^2+\beta^2)=0$$

represents two straight lines, and determine their point of intersection.

96. A is the point (1, 2) and P is a variable point on the line pair

$$2x^2+xy-6y^2-5x+11y-3=0;$$

show that the locus of the mid-point of AP has equation

$$4x^2+2xy-12y^2-11x+34y-20=0.$$

97. One of the lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

makes equal intercepts on the axes; show that

$$a+b=2h.$$

98. The line

$$gx+fy=0$$

meets the lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

at A, B; show that AB is bisected at the origin.

99. The equation

$$ax^2+2hxy+ay^2+2gx+2fy+c=0$$

represents two distinct lines; show that the origin lies on the bisector of one of the angles between these lines, if

$$g^2 = f^2.$$

100. Show that the line which is terminated by the lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and which is bisected at the point (x_1, y_1) has equation

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0.$$

101. The axes are rotated through the acute angle whose tangent is $\frac{3}{4}$; find the new coordinates of the points $(1, 2)$, $(5, -5)$, $(-4, 2)$.

102. Transform the equation

$$xy = c^2$$

the axes being rotated through 45°.

103. Transform the equation

$$xy - 2x + y - 6 = 0$$

the origin being moved to the point $(-1, 2)$ and the axes then rotated through 45°.

104. On changing the origin the line

$$2x - 3y + 1 = 0$$

becomes the line

$$2x - 3y + 6 = 0;$$

show that the new origin must lie on the line

$$2x - 3y - 5 = 0.$$

105. On rotating the axes through an angle θ, the line

$$2x + y - 3 = 0$$

becomes the line

$$x + 2y - 3 = 0;$$

show that $\tan\theta = -\frac{3}{4}$.

106. Transform the equation

$$x^2 + y^2 + 2x - 4y - 3 = 0$$

the origin being changed to the point $(-1, 2)$ and the axes then rotated through an angle θ.

107. Transform the equation

$$9x^2 - 24xy + 16y^2 - 14x - 23y + 36 = 0$$

the origin being changed to the point $(1, 1)$ and the axes then rotated through the acute angle whose tangent is $\frac{3}{4}$.

108. When the axes are rotated through an angle θ the equation

$$ax^2 + 2hxy + by^2 = 0$$

becomes

$$a_1x^2 + b_1y^2 = 0 ;$$

show that

$$\tan 2\theta = \frac{2h}{a-b}.$$

109. When the axes are rotated through an angle θ the equation

$$ax^2 + 2hxy + by^2 = 0$$

becomes

$$a_1x^2 + 2h_1xy + b_1y^2 = 0 ;$$

show that

$$a_1 + b_1 = a + b, \quad a_1b_1 - h_1{}^2 = ab - h^2.$$

110. Show that when the origin is changed the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is transformed into an equation of the form

$$ax^2 + 2hxy + by^2 + 2g_1x + 2f_1y + c_1 = 0.$$

PART II.—THE CIRCLE

CHAPTER IX

THE CIRCLE

§ **43.** *The equation of the circle whose centre is the origin and whose radius is a is*

$$x^2 + y^2 = a^2.$$

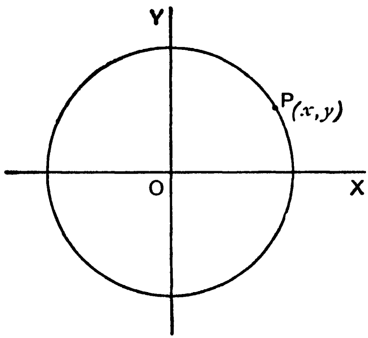

FIG. 36.

Let P (Fig. 36) be any point (x, y) on the circle.

Then $$OP^2 = x^2 + y^2 ;$$

$$\therefore\ x^2 + y^2 = a^2.$$

This equation is satisfied at any point on the circle and at no other point and is therefore the equation of the circle.

Note 1. The polar equation of the circle is $r = a$.

Note 2. In the case of oblique axes inclined at angle ω,

$$OP^2 = x^2 + y^2 + 2xy \cos \omega,$$

i.e. $$x^2 + y^2 + 2xy \cos \omega = a^2,$$

which is therefore the equation of the circle.

Example. Find the equation of the circle which has its centre at the origin and which passes through the point (x_1, y_1).

The equation is $\qquad x^2+y^2=a^2$

where $\qquad x_1{}^2+y_1{}^2=a^2$;

$\therefore$ the equation is $\qquad x^2+y^2=x_1{}^2+y_1{}^2$.

§ **44.** *The equation of the circle whose centre is the point* (h, k) *and whose radius is* a *is*

$$(x-h)^2+(y-k)^2=a^2.$$

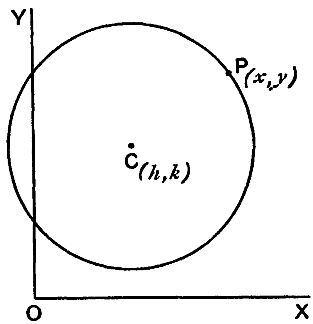

FIG. 37.

Let C (Fig. 37) be the point (h, k), and let $P(x, y)$ be any point on the circle.

Then $\qquad CP^2=(x-h)^2+(y-k)^2$,

i.e. $\qquad (x-h)^2+(y-k)^2=a^2$,

which is therefore the equation of the circle.

Note 1. This equation can be written

$$x^2+y^2-2hx-2ky+(h^2+k^2-a^2)=0;$$

hence the general equation of the second degree, viz.

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

cannot represent a circle unless $a=b$ and $h=0$.

Note 2. The polar equation of the circle may be obtained thus :

Let $P(r, \theta)$ (Fig. 38) be any point on the circle with centre $C(\rho, \alpha)$ and radius a.

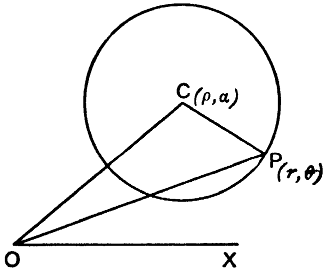

FIG. 38.

Then $$a^2 = CP^2$$
$$= r^2 + \rho^2 - 2r\rho \cos(\theta - \alpha),$$

i.e. $$r^2 - 2r\rho\cos(\theta - \alpha) = a^2 - \rho^2,$$

which is therefore the equation of the circle.

Note 3. In the case of oblique axes inclined at angle ω,

$$CP^2 \text{ (Fig. 37)} = (x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega,$$

i.e. $$(x-h)^2 + (y-k)^2 + 2(x-h)(y-k)\cos\omega = a^2,$$

which is therefore the equation of the circle.

§ **45.** *The equation*

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{.....................(i)}$$

represents a circle, centre $(-g, -f)$ *and radius* $\sqrt{g^2 + f^2 - c}$.

Equation (i) can be written

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c,$$

i.e. the distance of the point (x, y) from the fixed point $(-g, -f)$ is constant and $= \sqrt{g^2 + f^2 - c}$;

$\therefore$ the point (x, y) lies on a circle, centre $(-g, -f)$ and radius $\sqrt{g^2 + f^2 - c}$, i.e. equation (i) represents this circle.

Note 1. The equation

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

can be written $$x^2 + y^2 + 2\frac{G}{A}x + 2\frac{F}{A}y + \frac{C}{A} = 0,$$

and hence represents a circle, centre $\left(-\frac{G}{A}, -\frac{F}{A}\right)$ and radius

$$\sqrt{\left(\frac{G}{A}\right)^2+\left(\frac{F}{A}\right)^2-\frac{C}{A}}, \text{ i.e. } \frac{1}{A}\sqrt{G^2+F^2-AC}.$$

Note 2. If, in equation (i), $c=0$, the circle passes through the origin; if $f=0$, the circle has its centre on the x-axis, and if $g=0$ the centre is on the y-axis.

Example 1. *Find the centre and the length of the radius of the circle*

$$x^2+y^2+4x-6y-3=0.$$

The centre is the point $(-2, 3)$;

$$\text{length of radius}=\sqrt{2^2+(-3)^2-(-3)}=4.$$

Example 2. *Find the centre and the length of the radius of the circle*

$$9x^2+9y^2-12x+6y-4=0.$$

The centre is the point $(\frac{6}{9}, -\frac{3}{9})$, i.e. $(\frac{2}{3}, -\frac{1}{3})$;

$$\text{length of radius}=\tfrac{1}{9}\sqrt{(-6)^2+(3)^2-9(-4)}=1.$$

Example 3. *Find the equation of the circle passing through the points* $(-1, 3)$, $(2, 2)$, $(1, 4)$.

The equation is

$$x^2+y^2+2gx+2fy+c=0$$

where $1+9-2g+6f+c=0$ ....(i)

$4+4+4g+4f+c=0$ ....(ii)

and $1+13+2g+8f+c=0.$ ....(iii)

From (i) and (ii), $6g-2f-2=0,$

i.e. $3g-f-1=0.$ ....(iv)

From (ii) and (iii), $2g-4f-9=0.$ ....(v)

From (iv) and (v), $g=-\frac{1}{2}, f=-\frac{5}{2},$

$\therefore$ from (i), $c=4,$

and the required equation is

$$x^2+y^2-x-5y+4=0.$$

Example 4. *Find the equation of the circle whose centre is on the x-axis and which passes through the points* (0, 3), (4, 1).

The equation is $$x^2+y^2+2gx+c=0$$

where $$9+c=0,$$

and $$16+1+8g+c=0,$$

i.e. where $$c=-9,\ g=-1.$$

$\therefore$ the equation is $$x^2+y^2-2x-9=0.$$

§ **46.** *The circle which has as diameter the join of the points* $A(x_1, y_1)$, $B(x_2, y_2)$ *has equation*

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

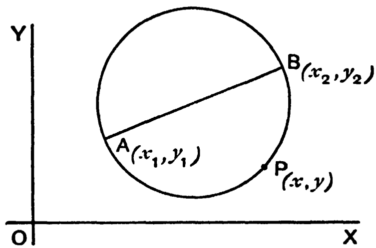

FIG. 39.

Let P (Fig. 39) be any point (x, y) on the circle.

Then the gradient of $$AP=\frac{y-y_1}{x-x_1},$$

and ,, ,, $$BP=\frac{y-y_2}{x-x_2},$$

but $$AP\perp BP;$$

$$\therefore\ \frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1,$$

i.e. $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0,$

which is therefore the equation of the circle.

Example. Find the intercept made on the x-axis by the circle which has as diameter the join of the points (0, -1), (2, 3).

The circle has equation

$$x(x-2)+(y+1)(y-3)=0,$$

i.e.
$$x^2+y^2-2x-2y-3=0.$$

Where $y=0$,
$$x^2-2x-3=0,$$

i.e.
$$x=-1, 3.$$

$\therefore$ the length of the intercept on the x-axis $=4$.

Exercises

1. Find the equation of the circle :

(i) with the origin as centre and of radius 2,

(ii) with the point (2, -3) as centre and of radius 3,

(iii) with the point (-1, 2) as centre and of radius $2\frac{1}{2}$,

(iv) with the origin as centre and passing through the point (1, 2),

(v) with the point (-1, 2) as centre and passing through the point (-2, 0),

(vi) with the points (4, 3), (2, -1) as ends of a diameter.

2. Find the centre and length of radius of the circle :

(i) $x^2+y^2=8$. (iv) $x^2+y^2+2x-4y+1=0$.

(ii) $2x^2+2y^2=3$. (v) $8x^2+8y^2-12x+20y-1=0$.

(iii) $(x-1)^2+(y+2)^2=5$, (vi) $(x+1)(x-3)+(y-2)(y-4)=0$.

3. Prove that the points (5, -14), (-10, 11), ($2\sqrt{13}$, 13) lie on a circle with the origin as centre ; find the equation of the circle.

4. The circle $x^2+y^2+2gx+2fy+c=0$

passes through the points (2, 1), (0, 5), (-1, 2) ; find g, f and c.

5. Find the equation of the circle passing through the points (-6, 5), (-3, -4), (2, 1) ; find the centre and length of radius of the circle.

6. Prove that the points (2, 0), (-1, 3), (-2, 0), (1, -1) lie on a circle ; determine its equation and the coordinates of its centre.

7. A, B are the points (-2, -3), (6, 1) ; C is the mid-point of AB and D divides AB internally in the ratio 3 : 1 ; find the coordinates of C and D and show that these points are concyclic with the points (2, 4), (4, 3).

8. A, B, C, D are the points $(-2, -4)$, $(1, 5)$, $(6, 0)$, $(3, 1)$; show that the circumradius of $\triangle ABC$ is equal in length to the join of the mid-points of AD and BC.

9. Find the points at which the line
$$x - 7y + 25 = 0$$
cuts the circle
$$x^2 + y^2 = 25.$$

10. Find the points at which the line
$$x + y - 4 = 0$$
cuts the circle
$$x^2 + y^2 + 4x - 2y - 20 = 0.$$

11. Find the length of the chord
$$3x - y + 5 = 0$$
of the circle
$$x^2 + y^2 = 5.$$

12. Find the length of the chord
$$4x - 3y - 5 = 0$$
of the circle
$$x^2 + y^2 + 3x - y - 10 = 0.$$

13. Show that the chord
$$x - 3y + 8 = 0$$
subtends a right angle at the centre of the circle
$$9x^2 + 9y^2 - 18x + 6y - 170 = 0.$$

14. Show that the chord
$$x - y + 3 = 0$$
of the circle
$$x^2 + y^2 + 3x - y = 0$$
is greater than the chord
$$x = 2y.$$

15. Find the equation of the circle having the origin as centre and passing through $P(4, -7)$; find the length of the chord PQ with gradient 2.

16. Show that $P(1, 1)$ lies on the circle
$$x^2 + y^2 + 4x + 6y - 12 = 0\,;$$
find the coordinates of the other extremity of the diameter through P.

17. Show that the circle
$$x^2 + y^2 - 2x + 2y - 3 = 0$$
trisects the line joining the points $(-4, -1)$, $(5, 2)$.

18. Show that the line joining the points $(-9, 3)$, $(3, -1)$ is divided internally and externally in the ratio $3:1$ by the circle

$$x^2+y^2-12x-6y=0.$$

19. Determine the ratios in which the circle

$$x^2+y^2-3x-y=0$$

divides the join of the points $(-1, -2)$, $(3, 6)$.

20. Show that the locus of the mid-points of lines drawn from the origin to the circle

$$x^2+y^2+2gx+2fy+4c=0$$

is the circle

$$x^2+y^2+gx+fy+c=0.$$

21. A point moves so that the square of its distance from the point $(-3, 2)$ varies as its distance from the line

$$3x-4y+2=0;$$

show that the locus is a circle, and find its equation given that the point $(3, 4)$ lies on the locus.

22. A, B are the points $(-2, 1)$, $(1, 2)$; P is a variable point such that

$$3AP=2BP;$$

show that P moves on a circle and determine the coordinates of the points at which the circle cuts AB.

23. Find the equation of the circle passing through the points $(-4, 1)$, $(3, 0)$ and having its centre on the y-axis.

24. Find the equation of the circle passing through the points $(0, 1)$, $(4, 3)$ and having its centre on the line

$$4x-5y-5=0.$$

25. Find the equation of the circle passing through the points $(-1, 3)$, $(3, 5)$ and having its centre on the line

$$x+2y-6=0.$$

26. Find the equations of the circles passing through the points $(4, -2)$, $(-3, -1)$ and having radius 5.

27. A, B, C are the points $(3, 5)$, $(-4, -2)$, $(3, -1)$; find the points which are concyclic with A, B, C and which lie on the line

$$x-3y+2=0.$$

28. Find the points of intersection of the circles

$$x^2+y^2+4x-2y-5=0, \qquad x^2+y^2+2x-7=0.$$

29. Find the points of intersection of the circles

$$x^2+y^2+x-3y-10=0, \quad 2x^2+2y^2-x-2y-15=0.$$

30. Show that the circles

$$x^2+y^2+2y-3=0, \quad x^2+y^2-6x-4y+3=0,$$

$$x^2+y^2+2x-6y-15=0$$

have a point in common.

31. Show that the circles

$$x^2+y^2-2x-3y=0, \quad x^2+y^2+x-y-6=0$$

intersect on the x- and y-axes.

32. Show that the centre of each of the circles

$$2x^2+2y^2-3x-4y+1=0, \quad 16x^2+16y^2-32x-1=0$$

lies on the other.

33. Show that the centres of the circles

$$x^2+y^2+2x-6y+9=0, \quad 2x^2+2y^2+3x+y-9=0$$

are the ends of a diameter of the circle

$$4x^2+4y^2+7x-11y=0.$$

34. The circle

$$x^2+y^2-5x-7y+6=0$$

meets the x-axis at A, B and the y-axis at C, D; show analytically that

$$OA \,.\, OB=OC \,.\, OD=6,$$

where O is the origin.

35. The circle

$$x^2+y^2-4x-y-12=0$$

cuts the x-axis at A, B and the y-axis at C, D; show analytically that

$$OA \,.\, OB=OC \,.\, OD,$$

where O is the origin.

36. Prove that the lines

$$x-7y-5=0, \;\ldots\ldots\text{(i)} \qquad x-y-5=0, \;\ldots\ldots\text{(ii)}$$
$$x-3y-7=0, \ldots\ldots\text{(iii)} \qquad x+3y+5=0 \;\ldots\ldots\text{(iv)}$$

are the sides, taken in order, of a cyclic quadrilateral.

37. Show that the lines

$$3x^2-4xy+y^2=0$$

intersect the lines

$$x^2-2xy+2x+6y-15=0$$

in concyclic points.

38. Show that the equation

$$\text{(i)}\quad x^2+y^2+2gx+2fy=0$$

represents a circle passing through the origin,

$$\text{(ii)}\qquad x^2+y^2+2gx=0$$

represents a circle passing through the origin and having its centre on the x-axis.

What form has the equation of a circle passing through the origin and having its centre on the y-axis ?

Investigate the locus of the equation

$$x^2+y^2+2gx-a^2=0.$$

39. Find the equation of the circle passing through the origin and making intercepts a and b on the x- and y-axes respectively.

40. A circle with its centre on the y-axis passes through the origin and the point (a, b) ; find its equation.

CHAPTER X

TANGENTS AND NORMALS

§ **47.** *Definition of Tangent and Normal.*

Let P, Q be two points on a curve and let P remain fixed while Q moves along the curve towards P; the line to which the secant PQ tends as Q tends to P is called the tangent to the curve at P. The straight line through P and perpendicular to the tangent at P is called the normal to the curve at P.

§ **48.** *The equation of the tangent to the circle*

$$x^2+y^2=a^2$$

at the point $P(x_1, y_1)$.

If $Q(x_2, y_2)$ be a second point on the circle, the equation of the line PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1),$$

but

$$x_1^2+y_1^2=a^2=x_2^2+y_2^2;$$

$$\therefore \quad y^2_2-y_1^2=-(x_2^2-x_1^2);$$

$$\therefore \quad \frac{y_2-y_1}{x_2-x_1}=-\frac{x_2+x_1}{y_2+y_1}.$$

$\therefore$ the equation of PQ is $y-y_1=-\dfrac{x_2+x_1}{y_2+y_1}(x-x_1)$.

Let Q now tend to P; the equation of the line to which PQ tends is

$$y-y_1=-\frac{2x_1}{2y_1}(x-x_1),$$

i.e.

$$xx_1+yy_1=x_1^2+y_1^2,$$

i.e.

$$xx_1+yy_1=a^2,$$

which is therefore the equation of the tangent at P.

Note 1. It follows that the tangent to the circle

$$Ax^2 + Ay^2 = a^2$$

at the point (x_1, y_1) has equation

$$Axx_1 + Ayy_1 = a^2.$$

Note 2. The foregoing method of deriving the equation of a tangent may be applied to curves other than the circle; in the particular case of the circle, the reader may readily obtain the equation of the tangent by finding the equation of the straight line perpendicular to a radius at its extremity.

Note 3. The reader familiar with the calculus may derive the equation of the tangent thus:

$$x^2 + y^2 = a^2;$$

$$\therefore\ 2x + 2y\frac{dy}{dx} = 0;$$

$$\therefore\ \frac{dy}{dx} = -\frac{x}{y};$$

$\therefore$ the tangent at (x_1, y_1) has equation

$$\frac{y - y_1}{x - x_1} = -\frac{x_1}{y_1},$$

i.e. $$xx_1 + yy_1 = x_1^2 + y_1^2,$$

i.e. $$xx_1 + yy_1 = a^2.$$

Example 1. *Determine the points at which the tangent at the point* (1, 2) *to the circle*

$$x^2 + y^2 = 5$$

cuts the circle

$$x^2 + y^2 = 10. \quad \text{.............................(i)}$$

The equation of the tangent is

$$x + 2y = 5. \quad \text{.............................(ii)}$$

Where (i) and (ii) intersect,

$$(5 - 2y)^2 + y^2 = 10,$$

i.e. $$y^2 - 4y + 3 = 0,$$

i.e. $$y = 1 \text{ or } 3,$$

and $\therefore$ $$x = 3 \text{ or } -1,$$

$\therefore$ the points of intersection are (3, 1), (−1, 3).

Example 2. *Show that a tangent from the point* $(\frac{1}{2}, \frac{3}{2})$ *to the circle*

$$4x^2 + 4y^2 = 5$$

touches the circle at the point $(-\frac{1}{2}, 1)$.

The point $(-\frac{1}{2}, 1)$ lies on the circle, for $4(-\frac{1}{2})^2 + 4(1)^2 = 5$, and the tangent at the point $(-\frac{1}{2}, 1)$ has equation

$$-2x + 4y = 5,$$

i.e. $$2x - 4y + 5 = 0;$$

and $$2(\tfrac{1}{2}) - 4(\tfrac{3}{2}) + 5 = 0;$$

$\therefore$ the tangent at $(-\frac{1}{2}, 1)$ passes through $(\frac{1}{2}, \frac{3}{2})$.

§ **49.** *The equation of the normal to the circle*

$$x^2 + y^2 = a^2$$

at the point $P(x_1, y_1)$.

The tangent at P has gradient $-\dfrac{x_1}{y_1}$;

$\therefore$ the normal ,, ,, ,, $\dfrac{y_1}{x_1}$;

$\therefore$ the normal has equation $y - y_1 = \dfrac{y_1}{x_1}(x - x_1)$,

i.e. $$\frac{y}{x} = \frac{y_1}{x_1}.$$

Note. In the particular case of the circle, the equation of the normal may be readily obtained by finding the equation of the radius whose extremity is the point (x_1, y_1).

§ **50.** *The condition that the line*

$$y = mx + c \quad \text{..........................(i)}$$

may touch the circle

$$x^2 + y^2 = a^2. \quad \text{................................(ii)}$$

Where (i) and (ii) meet,

$$x^2 + (mx + c)^2 = a^2,$$

i.e. $$(1 + m^2)x^2 + 2mcx + (c^2 - a^2) = 0. \quad \text{...............(iii)}$$

When the points of meeting are coincident, equation (iii) has equal roots,

and $\therefore$ $$m^2c^2=(1+m^2)(c^2-a^2),$$

i.e. $$c^2=a^2(1+m^2),$$

i.e. $$c=\pm a\sqrt{1+m^2},$$

which is the required condition.

Note. It follows that the lines

$$y=mx\pm a\sqrt{1+m^2}$$

are tangents to the circle (ii) for all values of m.

Example. Find the equations of the tangents to the circle

$$9x^2+9y^2=25,$$

which are parallel to the line

$$4x+3y+5=0.$$

The gradient of the tangents $=-\frac{4}{3}$;

$\therefore$ the equations of the tangents are

$$y=-\tfrac{4}{3}x\pm\tfrac{5}{3}\sqrt{1+(-\tfrac{4}{3})^2},$$

i.e. $$12x+9y-25=0 \quad \text{and} \quad 12x+9y+25=0.$$

§ 51. *The equation of the tangent to the circle*

$$x^2+y^2+2gx+2fy+c=0$$

at the point $P(x_1, y_1)$.

If $Q(x_2, y_2)$ be a second point on the circle, the equation of the line PQ is

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1),$$

but $x_1^2+y_1^2+2gx_1+2fy_1+c=0=x_2^2+y_2^2+2gx_2+2fy_2+c$;

$$\therefore\ (x_2^2-x_1^2)+(y_2^2-y_1^2)+2g(x_2-x_1)+2f(y_2-y_1)=0;$$

$$\therefore\quad (y_2-y_1)(y_2+y_1+2f)=-(x_2-x_1)(x_2+x_1+2g);$$

$\therefore$ the equation of PQ is

$$y-y_1=-\frac{x_2+x_1+2g}{y_2+y_1+2f}(x-x_1);$$

$\therefore$ the equation of the tangent at P is

$$y-y_1=-\frac{2(x_1+g)}{2(y_1+f)}(x-x_1),$$

i.e. $$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1,$$

i.e. $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=x_1^2+y_1^2+2gx_1+2fy_1+c,$

i.e. $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$

Note 1. It follows that the equation of the tangent at the point (x_1, y_1) on the circle $$Ax^2+Ay^2+2Gx+2Fy+C=0$$

is $$Axx_1+Ayy_1+G(x+x_1)+F(y+y_1)+C=0.$$

Note 2. The reader familiar with the calculus may derive the equation of the tangent thus:

$$x^2+y^2+2gx+2fy+c=0\ ;$$

$$\therefore\quad 2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0\ ;$$

$$\therefore\quad \frac{dy}{dx}=-\frac{x+g}{y+f};$$

$\therefore$ the tangent at (x_1, y_1) has equation

$$y-y_1=-\frac{x_1+g}{y_1+f}(x-x_1),$$

i.e. $$xx_1+yy_1+gx+fy=x_1^2+y_1^2+gx_1+fy_1,$$

i.e. $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=x_1^2+y_1^2+2gx_1+2fy_1+c,$

i.e. $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$

Note 3. The tangent at (x_1, y_1) has gradient $-\dfrac{x_1+g}{y_1+f}$;

$\therefore$ the normal ,, ,, ,, ,, $\dfrac{y_1+f}{x_1+g}$;

$\therefore$ the normal at (x_1, y_1) has equation

$$\frac{y-y_1}{x-x_1}=\frac{y_1+f}{x_1+g}.$$

Example 1. *Find the equation of the tangent to the circle*

$$x^2+y^2+4x-8y-5=0$$

at the point (2, 7).

The required equation is

$$2x+7y+2(x+2)-4(y+7)-5=0,$$

i.e. $$4x+3y-29=0.$$

Example 2. Show that the line

$$2x + y + 2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

is a tangent to the circle

$$x^2 + y^2 + 6x + 2y + 5 = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Where (i) and (ii) meet

$$x^2 + (-2x - 2)^2 + 6x + 2(-2x - 2) + 5 = 0,$$

i.e. $$x^2 + 2x + 1 = 0,$$

and this equation has equal roots; therefore line (i) is a tangent.

Example 3. Find the equation of the circle which passes through the point (1, 0) *and touches the line*

$$3x + 2y - 4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

at the point (2, −1).

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

Since (1, 0) lies on the circle,

$$1 + 2g + c = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

The equation of the tangent at (2, −1) is

$$2x - y + g(x + 2) + f(y - 1) + c = 0,$$

i.e. $$(g + 2)x + (f - 1)y + (2g - f + c) = 0. \quad \ldots\ldots\ldots\ldots\text{(iii)}$$

∴ from (i) and (iii),

$$\frac{g+2}{3} = \frac{f-1}{2} = \frac{2g - f + c}{-4},$$

which, from (ii), $$= \frac{-1 - f}{-4};$$

$$\therefore f = 3, \quad g = 1, \quad c = -3,$$

and the required equation is

$$x^2 + y^2 + 2x + 6y - 3 = 0.$$

§ **52.** *The equation of the chord of contact of tangents from the point* $P(x_1, y_1)$ *to the circle*

$$x^2+y^2=a^2.$$

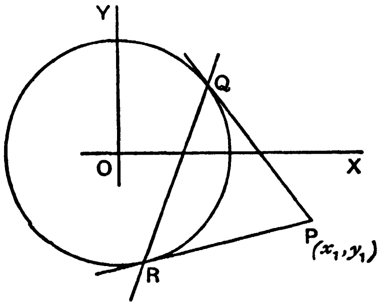

FIG. 40.

Let the tangents from P (Fig. 40) touch the circle at Q and R. If Q is the point (x_2, y_2), PQ has the equation

$$xx_2+yy_2=a^2\ ;$$

and P lies on this line,

$$\therefore\quad x_1x_2+y_1y_2=a^2,$$

i.e. Q lies on the line

$$xx_1+yy_1=a^2.$$

Similarly R lies on this line, which is therefore the line QR, i.e. the equation of the chord of contact is

$$xx_1+yy_1=a^2.$$

Note. In the same way it is proved that the equation of the chord of contact of tangents from the point (x_1, y_1) to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is
$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

Example. Find the equation of the tangents from the origin to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

The chord of contact of tangents from the origin has equation

$$gx+fy+c=0\ ;$$

$\therefore$ the equation of the tangents is

$$x^2+y^2+(2gx+2fy)\left(-\frac{gx+fy}{c}\right)+c\left(-\frac{gx+fy}{c}\right)^2=0,$$

i.e. $$c(x^2+y^2)=(gx+fy)^2.$$

§ **53.** *The equation of the tangents from the point $P(x_1, y_1)$ to the circle*

$$x^2+y^2=a^2.$$

We give two methods of finding the tangents, as both methods are instructive.

1*st Method.* Change the origin to P and the equation of the circle becomes

$$(x+x_1)^2+(y+y_1)^2=a^2,$$

i.e. $$x^2+y^2+2x_1x+2y_1y+(x_1^2+y_1^2-a^2)=0.$$

Referred to the new axes, the tangents from P (see § 52, Example) have therefore the equation

$$(x_1^2+y_1^2-a^2)(x^2+y^2)=(x_1x+y_1y)^2.$$

Referred to the original axes the equation is therefore

$$(x_1^2+y_1^2-a^2)\{(x-x_1)^2+(y-y_1)^2\}=\{x_1(x-x_1)+y_1(y-y_1)\}^2,$$

i.e. $$(x_1^2+y_1^2-a^2)\{(x^2+y^2-a^2)+(x_1^2+y_1^2-a^2)-2(xx_1+yy_1-a^2)\}$$
$$=\{(xx_1+yy_1-a^2)-(x_1^2+y_1^2-a^2)\}^2,$$

i.e. $$(x_1^2+y_1^2-a^2)(x^2+y^2-a^2)=(xx_1+yy_1-a^2)^2.$$

2*nd Method.* Join P (Fig. 41) to any point Q (x, y).

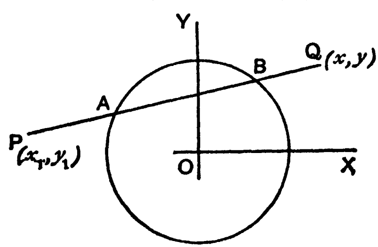

FIG. 41.

The point dividing PQ in the ratio $k:1$ lies on the circle if

$$\left(\frac{kx+x_1}{k+1}\right)^2+\left(\frac{ky+y_1}{k+1}\right)^2=a^2,$$

i.e. if $(x^2+y^2-a^2)k^2+2(xx_1+yy_1-a^2)k+(x_1{}^2+y_1{}^2-a^2)=0.$ (i)

If PQ cuts the circle at A and B, the two values of k given by equation (i) are the values of the ratios $PA:AQ$ and $PB:BQ$. Now if Q lies on either of the tangents from P to the circle, these two values of k must be equal and $\therefore$

$$(x_1{}^2+y_1{}^2-a^2)(x^2+y^2-a^2)=(xx_1+yy_1-a^2)^2,$$

which is therefore the equation of the tangents.

Note. Using either of the foregoing methods, the reader should prove that the equation of the tangents from the point (x_1, y_1) to the circle

$$x^2+y^2+2gx+2fy+c=0$$

is $\quad (x_1{}^2+y_1{}^2+2gx_1+2fy_1+c)(x^2+y^2+2gx+2fy+c)$

$$=\{xx_1+yy_1+g(x+x_1)+f(y_1+y_1)+c\}^2.$$

Example. Find the equations of the tangents from the point $(-3, -4)$ *to the circle*

$$x^2+y^2-4x-2y-5=0.$$

The equation of the pair of tangents is

$$(9+16+12+8-5)(x^2+y^2-4x-2y-5)$$
$$=\{-3x-4y-2(x-3)-(y-4)-5\}^2,$$

i.e. $\quad 40(x^2+y^2-4x-2y-5)=(-5x-5y+5)^2,$

i.e. $\quad 8(x^2+y^2-4x-2y-5)=5(x^2+2xy+y^2-2x-2y+1),$

i.e. $\quad 3x^2-10xy+3y^2-22x-6y-45=0$

i.e. $\quad (3x-y+5)(x-3y-9)=0;$

$\therefore$ the equations of the tangents are

$$3x-y+5=0, \quad x-3y-9=0.$$

Exercises

1. Find the equations of the tangents to the following circles at the points indicated:

(i) $x^2+y^2=25$, $(-3, 4)$,
(ii) $8x^2+8y^2=5$, $(\frac{3}{4}, -\frac{1}{4})$,
(iii) $4x^2+4y^2=25$, $(-2, -\frac{3}{2})$,
(iv) $x^2+y^2+4x+6y=0$, $(0, 0)$,
(v) $x^2+y^2-6x-2y-3=0$, $(5, 4)$,
(vi) $(x-1)^2+(y+2)^2=5$, $(3, -3)$,
(vii) $2x^2+2y^2-3x+4y+1=0$, $(1, -2)$.

2. Find the equations of the normals to the circles of Ex. 1 at the points indicated.

3. Show that each of the following lines is a tangent to the circle indicated, and find the coordinates of the point of contact:

(i) $3x-4y=25$, $\quad x^2+y^2=25$,
(ii) $5x+y=0$, $\quad x^2+y^2-10x-2y=0$,
(iii) $2x+y-10=0$, $\quad x^2+y^2-4x-2y=0$,
(iv) $x+2y-12=0$, $\quad x^2+y^2-6x-4y+8=0$,
(v) $8x-7y+22=0$, $\quad 3x^2+3y^2-2x-5y-7=0$.

4. Find the equation of the circle with centre $(1, 2)$ and touching the line
$$4x-3y+12=0.$$

5. The line
$$2x+3y-5=0$$
is a tangent to a circle with centre $(3, 4)$; find the intercept which the circle makes on the y-axis.

6. Show that the tangents to the circle
$$x^2+y^2+6x-2y+5=0$$
at the points $P(-1, 2)$, $Q(-2, -1)$ meet at the origin O; show that $P\hat{O}Q$ is a right angle, and prove analytically that the bisector of $P\hat{O}Q$ passes through the centre of the circle.

7. Show that the tangent to the circle
$$x^2+y^2-6x+2y+5=0$$
at the point $(1, 0)$ touches the circle
$$5x^2+5y^2=4.$$

8. Show that the tangent to the circle
$$x^2+y^2=13$$
at the point $(-2, 3)$ touches the circle
$$x^2+y^2-10x+2y-26=0.$$

9. Show that the straight line joining the centres of the circles

$$x^2+y^2-4y-12=0, \quad x^2+y^2-4x+8y-5=0$$

touches the circle

$$2x^2+2y^2-8x-4y+5=0.$$

10. Show that the circles

$$x^2+y^2=13, \quad x^2+y^2-12x-8y+39=0$$

touch at the point (3, 2).

11. Show that the circles

$$x^2+y^2+2ax+4ay-3a^2=0, \quad x^2+y^2-8ax-6ay+7a^2=0$$

touch, and determine the point of contact.

12. Show that the circles

$$x^2+y^2+8x+14=0, \quad x^2+y^2-4x-8y-30=0$$

intersect at the point $(-3, -1)$ and that the radius of either circle through this point is a tangent to the other circle.

13. Show that the circles

$$x^2+y^2+6x-11=0, \quad x^2+y^2-4x-1=0$$

intersect at the point (1, 2) and that their tangents at this point are perpendicular to each other.

14. Show that the tangents to the circles

$$x^2+y^2-6ax+6ay+16a^2=0, \quad x^2+y^2-2ax+6ay+8a^2=0$$

at either of their points of intersection are perpendicular to each other.

15. The circle

$$x^2+y^2-4x-6y+a=0$$

touches the x-axis; find a and the coordinates of the point of contact.

16. In each of the following examples, the given line touches the given circle; find k.

(i) $2x-y+k=0, \quad x^2+y^2=5,$

(ii) $2x-ky-13=0, \quad x^2+y^2=13,$

(iii) $kx+3y-5=0, \quad 2x^2+2y^2=5,$

(iv) $y=kx, \quad x^2+y^2-4x+2y=0,$

(v) $2x-ky-3=0, \quad x^2+y^2+4x-4y-5=0.$

(vi) $x+3y+k=0, \quad 2x^2+2y^2+4x+8y+5=0,$

(vii) $x-ky-7a=0, \quad x^2+y^2+4ax+4ay+3a^2=0,$

(viii) $2x-y+k=0, \quad x^2+y^2+6ax+4ay+8a^2=0.$

17. Show that the line

$$lx + my + n = 0$$

touches the circle

$$(x - h)^2 + (y - k)^2 = r^2,$$

if $$(hl + km + n)^2 = r^2(l^2 + m^2).$$

18. A, B are the points at which the lines

$$y = \pm a$$

cut any tangent to the circle

$$x^2 + y^2 = a^2 ;$$

show that AB subtends a right angle at the origin.

19. Show that the equation

$$x^2 + y^2 - 2ax = 0$$

represents a circle touching the y-axis at the origin, and that the equation

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

represents a circle touching the x- and y-axes.

20. Find the equations of the tangents to the circle

$$x^2 + y^2 + 4x - 8y + 2 = 0,$$

which make on the axes intercepts equal in magnitude and sign.

21. The line $$2x - 3y - 9 = 0$$

touches the circles

$$x^2 + y^2 - 2x - 4y - a = 0, \quad x^2 + y^2 - 8x - 8y + b = 0 ;$$

find a and b.

22. The tangent to the circle

$$x^2 + y^2 = 25$$

at the point (4, 3) meets the circle

$$x^2 + y^2 = 50$$

at P and Q; show that the tangents to the second circle at P and Q are perpendicular to each other.

23. The tangent at the point (-2, 1) on the circle

$$x^2 + y^2 = 5$$

meets the circle

$$x^2 + y^2 - 6x - 12y + 35 = 0$$

at P and Q; show that the tangents to the second circle at P and Q are perpendicular to each other.

24. Find the equations of the tangents to the circle

$$x^2 + y^2 = 25,$$

which are parallel to the line

$$4x - 3y - 2 = 0.$$

25. Find the equations of the tangents to the circle

$$x^2 + y^2 - 6x + 10y + 29 = 0,$$

which are parallel to the line

$$2x + y + 8 = 0.$$

26. Show that the line

$$4x - 3y - 14 = 0$$

is a tangent to the circle

$$x^2 + y^2 - 4x - 6y + 4 = 0,$$

and determine the equation of the other tangent having the same gradient.

27. Find the equations of the tangents to the circle

$$x^2 + y^2 - 6x + 4y + 8 = 0,$$

which are perpendicular to the line

$$2x - y - 1 = 0.$$

28. Find the equation of the circle which passes through the point (2, 3) and touches the line

$$2x - 3y - 13 = 0$$

at the point $(2, -3)$.

29. Find the equations of the circles which touch the x-axis at the point (4, 0) and make an intercept of 6 on the y-axis.

30. Find the equations of the circles which touch the line

$$x = 3$$

at the point (3, 0) and which make an intercept of 6 on the line

$$y + 4 = 0.$$

31. Show that the lines

$$2x^2 - 3xy - 2y^2 = 0$$

are tangents from the origin to the circle

$$x^2 + y^2 - 2x - 6y + 5 = 0.$$

32. Tangents are drawn from the given points to the given circles; find the equations of the chords of contact and the coordinates of the points of contact:

(i) $(-1, 5)$, $x^2 + y^2 = 13$,
(ii) $(1, 2)$, $x^2 + y^2 - 4x + 6y = 0$,
(iii) $(-1, -2)$, $x^2 + y^2 + 4x + 10y + 24 = 0$,
(iv) $(-3, 1)$. $4x^2 + 4y^2 + 4x + 2y - 5 = 0$.

33. Tangents are drawn from the points (2, 9), (3, 7), (5, 3) to the circle

$$x^2+y^2-2x-4y-4=0\ ;$$

show that the chords of contact are concurrent.

34. Write down the equation of the chord of contact of tangents from the point (1, 7) to the circle

$$x^2+y^2=25\ ;$$

hence, or otherwise, find the equations of the tangents.

35. Write down the equation of the chord of contact of tangents from the point $(-3,\ -2)$ to the circle

$$x^2+y^2-4x-6y+8=0\ ;$$

hence, or otherwise, find the equations of the tangents.

36. Find the equation of the tangents from the origin to the circle:

(i) $x^2+y^2-4x+6y+4=0$,

(ii) $x^2+y^2-2ax+4ay+4a^2=0$,

(iii) $x^2+y^2-2ax\sec\theta-2ay\operatorname{cosec}\theta+a^2(\tan^2\theta+\operatorname{cosec}^2\theta)=0$.

37. Find the equation of the tangents from the given point to the given circle:

(i) $(-5, 5)$, $x^2+y^2=5$,

(ii) $(2, 4)$, $x^2+y^2+x-3y=0$,

(iii) $(3, 2)$, $x^2+y^2+4x+6y+8=0$.

38. Show that the tangents from the point (4, 3) to the circle

$$x^2+y^2-4x-4y+6=0$$

touch the circle

$$x^2+y^2+8x+2y-15=0.$$

39. Tangents are drawn from the origin and the point (g, f) to the circle

$$x^2+y^2+2gx+2fy+c=0\ ;$$

show that the chords of contact are parallel and that the distance between them is

$$\frac{r^2}{2\sqrt{r^2+c}},$$

where r is the radius of the circle.

40. Show that the tangents from the point (g, f) to the circle

$$x^2+y^2+2gx+2fy-c=0$$

are perpendicular to each other if

$$c=g^2+f^2.$$

CHAPTER XI

POLES AND POLARS

§ **54.** *Definition of Polar and Pole.*

The polar of a point P with respect to a circle is the locus of the point of intersection of tangents drawn at the extremities of a variable chord passing through P.

The point P is called the pole of the locus.

§ **55.** *The polar of the point $P(x_1, y_1)$ with respect to the circle*

$$x^2 + y^2 = a^2$$

is the straight line $$xx_1 + yy_1 = a^2.$$

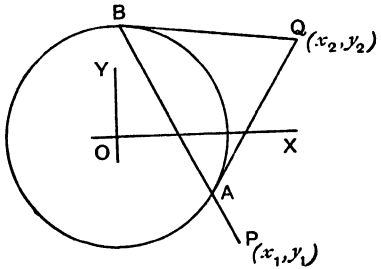

FIG. 42.

Let AB (Fig. 42), a chord of the circle, pass through P, and let the tangents at A and B intersect at $Q(x_2, y_2)$.

Then AB has equation

$$xx_2 + yy_2 = a^2;$$

P lies on this line,

$$\therefore\ x_1x_2 + y_1y_2 = a^2,$$

i.e. Q lies on the line

$$xx_1 + yy_1 = a^2,$$

which is therefore the polar of P.

Note 1. P need not lie outside the circle; it may lie inside or on the circle.

Note 2. The polar has gradient $-\frac{x_1}{y_1}$ and is therefore perpendicular to the line joining P to the centre of the circle.

Note 3. Using the method of this paragraph, the reader should prove that the polar of the point (x_1, y_1) with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

has equation

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0.$$

Note 4. When P lies outside the circle, the chord of contact of tangents from P is the polar of P.

§ **56.** *If the polar of P passes through Q, the polar of Q passes through P.*

Consider the circle

$$x^2+y^2=a^2,$$

and let P, Q be the points (x_1, y_1), (x_2, y_2).

Then the polar of P has equation

$$xx_1+yy_1=a^2,$$

and Q lies on this line,

$$\therefore \quad x_2x_1+y_2y_1=a^2,$$

i.e. P lies on the line

$$xx_2+yy_2=a^2,$$

which is the polar of Q.

Note. Points such that the polar of either with respect to a circle passes through the other are called conjugate points, and lines such that either passes through the pole of the other are called conjugate lines with respect to the circle.

Example 1. *Find the equation of the polar of the point* $(2, -5)$ *with respect to the circle*

$$4x^2+4y^2+20x-12y+9=0.$$

The polar has equation

$$8x-20y+10(x+2)-6(y-5)+9=0,$$

i.e.

$$18x-26y+59=0.$$

Example 2. *Show that the lines*

$$x+7y-11=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and

$$3x+y-2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

are conjugate with respect to the circle

$$x^2+y^2-6x-6y+5=0.$$

Let the pole of (i) be $P(x_1, y_1)$; then the polar of P has equation

$$xx_1+yy_1-3(x+x_1)-3(y+y_1)+5=0,$$

i.e.

$$(x_1-3)x+(y_1-3)y-(3x_1+3y_1-5)=0,$$

and $\therefore$ from (i),

$$\frac{x_1-3}{1}=\frac{y_1-3}{7}=\frac{3x_1+3y_1-5}{11},$$

i.e.

$$7x_1-y_1-18=0,$$

and

$$8x_1-3y_1-28=0,$$

whence

$$x_1=2, \quad y_1=-4.$$

$\therefore$ P is the point $(2, -4)$, and, since

$$3(2)+(-4)-2=0,$$

P lies on (ii), and therefore (i) and (ii) are conjugate lines.

§ **57.** *If the polar of P with respect to a circle, centre C and radius a, meets the radius through P at Q,*

$$CP \,.\, CQ=a^2.$$

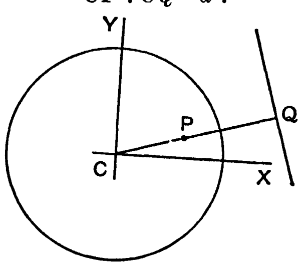

Fig. 43.

Let the circle have equation

$$x^2+y^2=a^2,$$

and let P (Fig. 43) be the point (x_1, y_1).

Then the polar of P has equation

$$xx_1 + yy_1 = a^2;$$

$$\therefore \qquad CQ = \frac{a^2}{\sqrt{x_1^2 + y_1^2}},$$

but
$$CP = \sqrt{x_1^2 + y_1^2};$$

$$\therefore \quad CP \,.\, CQ = a^2.$$

Note 1. Points P and Q lying on a line through the centre C of a circle of radius a and such that $CP \,.\, CQ = a^2$ are called inverse points with respect to the circle.

Note 2. The polar of a point P with respect to a circle, centre C, may be defined as the line perpendicular to CP and passing through the point inverse to P.

It is left to the reader to derive from this definition the equation of the polar of $P\ (x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$.

§ **58.** *If AB, a chord of a circle, passes through a point P and intersects the polar of P at Q, (AB, PQ) is harmonic.*

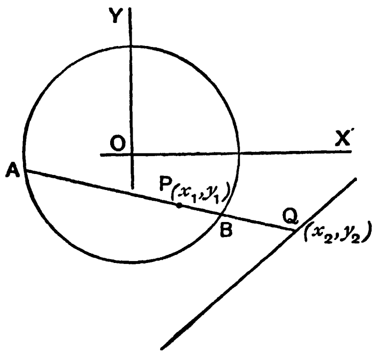

FIG. 44.

Let P, Q (Fig. 44) be the points (x_1, y_1), (x_2, y_2), and let the circle have equation

$$x^2 + y^2 = a^2.$$

The point dividing PQ in the ratio $k : 1$ has coordinates

$$\frac{kx_2 + x_1}{k+1}, \quad \frac{ky_2 + y_1}{k+1},$$

and lies on the circle if

$$(kx_2 + x_1)^2 + (ky_2 + y_1)^2 = a^2 (k+1)^2,$$

i.e. if $(x_2{}^2 + y_2{}^2 - a^2)k^2 + 2(x_1x_2 + y_1y_2 - a^2)k + (x_1{}^2 + y_1{}^2 - a^2) = 0,$

and, since Q lies on the polar of P, the coefficient of k in this equation is zero, and therefore the two values of k are equal in magnitude and opposite in sign, i.e. A, B divide PQ internally and externally in the same ratio ;

$$\therefore\ (PQ, AB) \text{ is harmonic ;}$$

$$\therefore\ (AB, PQ) \text{ is harmonic.}$$

Note. If AB is a variable chord through P, the locus of Q the harmonic conjugate of P with respect to A and B is the polar of P.

Exercises

1. Find the equation of the polar of each of the following points with respect to the given circles :

(i) $(-1, 5)$, $x^2 + y^2 - 6x - 2y + 1 = 0$,
(ii) $(-2, -4)$, $x^2 + y^2 - 8x + 4y + 9 = 0$,
(iii) $(3, -2)$, $3x^2 + 3y^2 - 6x + 4y + 4 = 0$,
(iv) (g, f), $x^2 + y^2 + 2gx + 2fy + c = 0$,
(v) $\left(-\dfrac{a^2m}{c}, \dfrac{a^2}{c}\right)$, $x^2 + y^2 = a^2$.

2. Find the poles of the following lines with respect to the given circles :

(i) $4x + y = 5$, $x^2 + y^2 = 5$,
(ii) $8x + 20y + 25 = 0$, $4x^2 + 4y^2 = 25$,
(iii) $3x + 4y + 2 = 0$, $x^2 + y^2 - 6x - 2y - 5 = 0$,
(iv) $4x + 3y - 7 = 0$, $x^2 + y^2 + 8x - 7 = 0$,
(v) $x + 3y - 1 = 0$, $x^2 + y^2 - 4x - 12y + 2 = 0$.

3. Show that the points (3, 4) and (9, 12) are conjugate with respect to the circle

$$x^2 + y^2 - 4x - 6y - 3 = 0.$$

4. Show that the points (2, 4) and $(-5, 3)$ are conjugate with respect to the circle

$$x^2 + y^2 - 5x - 3y + 1 = 0.$$

5. Show that the polar of each of the points $(1, 2)$, $(7, 9)$, $(-35, 30)$ with respect to the circle

$$x^2+y^2=25$$

passes through the other two given points.

6. The sides of a triangle have equations

$$x+y=8, \quad x+3y=4, \quad 5x-y=4\,;$$

show that each side is the polar of the opposite vertex with respect to the circle

$$x^2+y^2=8.$$

7. Show that the lines

$$3x-y-25=0, \quad 7x-4y-25=0$$

are conjugate with respect to the circle

$$x^2+y^2=25.$$

8. Show that the lines

$$x+5y+2=0, \quad x-2y-9=0$$

are conjugate with respect to the circle

$$x^2+y^2-4x-2y-4=0.$$

9. The polar of the point (a, b) with respect to the circle

$$x^2+y^2+4x-2y+1=0$$

passes through the points $(-6, -1)$, $(-3, 2)$; find a and b.

10. The polar of the point (g, f) with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

passes through the origin; show that

$$g^2+f^2+c=0.$$

11. Show that, for all values of t, the polar of the point $(2t, t-4)$ with respect to the circle

$$x^2+y^2-4x-6y+1=0$$

passes through the point $(3, 1)$.

12. Prove that the polar of the point $(a, -2a)$ with respect to the circle

$$x^2+y^2+2ax+4ay+4a^2=0$$

touches the circle

$$4x^2+4y^2+8ax+16ay+19a^2=0.$$

13. The chord of contact of tangents from the point P to the circle

$$x^2+y^2+2x-8=0$$

passes through the point $(2, 3)$; find the locus of P.

14. Prove that the polars of the points $(-1, 2)$, $(7, -7)$ with respect to the circle

$$2x^2+2y^2-3x+4y-1=0$$

intersect on the line

$$x-8y+5=0.$$

15. The points (x_1, y_1), (x_2, y_2) lie on the line

$$y=mx+c;$$

prove that their polars with respect to the circle

$$x^2+y^2=a^2$$

intersect at the point $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

16. Determine the point of intersection of the polars of the point $(1, 3)$ with respect to the circles

$$3x^2+3y^2=5, \quad x^2+y^2+4x-6y+8=0.$$

17. The equation of two chords of the circle

$$2x^2+2y^2=5$$

is

$$(4x+2y-5)(4x-2y-5)=0;$$

show that the tangents in pairs from the extremities of the chords intersect on the line

$$x=2.$$

18. Prove that the poles of the lines

$$y+2=0, \quad 2x+7y+8=0, \quad 2x-9y-24=0$$

with respect to the circle

$$x^2+y^2-4x+8y+4=0$$

are collinear, and find the equation of the line on which they lie.

19. Show that the polars of the points $(2, 0)$, $(7, -3)$, $(-3, 3)$ with respect to the circle

$$3x^2+3y^2-2y-4=0$$

are concurrent, and find the point of concurrence.

20. The equation of two chords OA, OB of the circle

$$x^2+y^2+10x=0$$

is

$$x^2-xy-6y^2=0;$$

find the pole of AB with respect to the circle.

21. Find the poles of the lines

$$6x^2+13xy+6y^2=0$$

with respect to the circle

$$x^2+y^2-2x-2y-3=0.$$

22. P, Q are the points (3, 4), (5, 12); PR, QS are perpendiculars to the polars of Q and P respectively, with respect to the circle

$$x^2+y^2=5;$$

show that $OP:OQ=PR:QS$, where O is the origin.

23. P, Q are any two points; PR is the perpendicular from P to the polar of Q with respect to any circle, and QS is the perpendicular from Q to the polar of P with respect to the same circle; prove that $OP:OQ=PR:QS$, where O is the centre of the circle.

24. A line through $P(-4, 3)$ has equation

$$x+3y-5=0$$

and intersects the circle

$$x^2+y^2=5$$

at A, B; the polar of P intersects the given line at Q; show that $PA=\pm 3AQ$.

25. The line through $P(1, m)$, parallel to the x-axis, intersects the circle

$$x^2+y^2=a^2$$

at A, B; the polar of P intersects the given line at Q; show that

$$AQ^2=(a^2-m^2)PA^2.$$

26. Prove that the polar of the origin with respect to the circle

$$x^2+y^2+2x-6y-6=0$$

touches the circle

$$5x^2+5y^2=18.$$

27. The polar of the origin with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

touches the circle

$$x^2+y^2=a^2;$$

show that

$$c^2=a^2(f^2+g^2).$$

28. Prove that the poles of the lines

$$ax^2+2hxy+by^2=0$$

with respect to the circle

$$x^2+y^2+2gx+2fy+c=0$$

lie on the line

$$gx+fy+c=0.$$

29. Prove that the poles of the lines

$$a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2=0$$

with respect to the circle $\quad x^2+y^2=a^2$

lie on the line $\quad px+qy=a^2.$

30. The tangent at the point $(2, -1)$ to the circle

$$x^2+y^2+2x-6y-15=0$$

meets the line

$$x+7y+30=0$$

at Q; find the coordinates of Q, prove that the polar of Q is parallel to the given line and determine the equation of the other tangent from Q.

31. The points (α, β), (x, y) are inverse with respect to the circle

$$x^2+y^2=a^2;$$

show that

$$x=\frac{a^2\alpha}{\alpha^2+\beta^2}, \quad y=\frac{a^2\beta}{\alpha^2+\beta^2}.$$

32. Show that the points $(1, -2)$, $(2, -4)$ are inverse with respect to the circle

$$x^2+y^2=10.$$

33. The points (α, β), (x, y) are inverse with respect to the circle

$$x^2+y^2+2gx+2fy+c=0;$$

show that

$$\frac{\alpha+g}{x+g}=\frac{\beta+f}{y+f}=\frac{g^2+f^2-c}{(x+g)^2+(y+f)^2}.$$

34. P, Q are inverse points with respect to the circle

$$x^2+y^2=10;$$

P lies on the circle

$$x^2+y^2-4x-2y=0;$$

show that the locus of Q has equation

$$2x+y-5=0.$$

35. P, Q are inverse points with respect to the circle

$$x^2+y^2=1;$$

P lies on the circle

$$x^2+y^2+2x+2y=0;$$

find the locus of Q.

36. P, Q are inverse points with respect to the circle

$$x^2+y^2=4;$$

P lies on the line

$$x-2y+1=0;$$

show that the locus of Q is the circle

$$x^2+y^2+4x-8y=0.$$

37. P is any point on the line

$$x \cos \alpha + y \sin \alpha = p ;$$

Q is the inverse of P with respect to the circle whose centre is the origin and whose radius is p; show that the locus of Q is the circle

$$x^2 + y^2 = p(x \cos \alpha + y \sin \alpha).$$

38. P, Q are inverse points with respect to the circle

$$x^2 + y^2 = 3 ;$$

P lies on the line

$$3x - y + 1 = 0 ;$$

find the locus of Q.

39. P is any point on the circle

$$x^2 + y^2 - 2x - 6y + 9 = 0 ;$$

find the locus of the inverse of P with respect to the circle

$$x^2 + y^2 - 6x - 4y + 9 = 0.$$

40. P is any point on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 ;$$

show that the locus of the inverse of P with respect to the circle

$$x^2 + y^2 = a^2$$

is the circle

$$c(x^2 + y^2) + 2a^2(gx + fy) + a^4 = 0.$$

CHAPTER XII

POWER. RADICAL AXIS. COAXAL CIRCLES. ORTHOGONAL CIRCLES

§ **59.** *Definition of Power.*

The power of a point P with respect to a circle is $PA \cdot PB$ where A and B are the points at which any line through P meets the circle; $PA \cdot PB$ is considered positive or negative according as P is without or within the circle.

Note. If P is on the circle, its power is zero, for one of the points A, B coincides with P.

§ **60.** *The power of* $P(x_1, y_1)$ *with respect to the circle*

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{.....................(i)}$$

is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

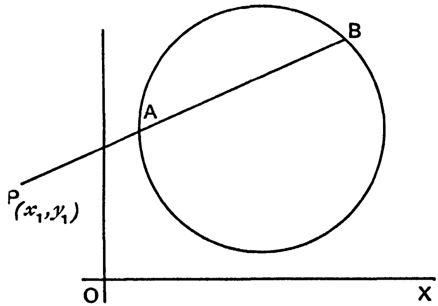

FIG. 45.

Let PAB (Fig. 45) be inclined at angle θ to OX; then the equation of PAB is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r,$$

and this line meets the circle (i) where

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta)$$
$$+ 2f(y_1 + r\sin\theta) + c = 0,$$

i.e. where

$$r^2 + 2r(\overline{x_1 + g}\cos\theta + \overline{y_1 + f}\sin\theta)$$
$$+ (x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c) = 0. \quad \text{..................(ii)}$$

The values of r given by this equation are the algebraic measures of the lines PA and PB, and will have opposite signs if P lies between A and B and the same sign if it does not; therefore the product of the roots of (ii), viz.

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c,$$

is positive or negative according as P lies without or within the circle, and therefore the power of P is

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

Note 1. We have proved, incidentally, that if P is any point in the plane of a circle and lines PAB are drawn to meet the circle at A and B,

$$PA \,.\, PB = \text{a constant},$$

for the expression obtained for $PA \,.\, PB$ is independent of θ, the inclination of PAB. It follows that, when P is outside the circle, $PA \,.\, PB =$ the square on the tangent from P.

Note 2. The power of P is sometimes defined as the square on the tangent from P.

Note 3. The significance of the constant c in equation (i) is now evident; c is the power of the origin with respect to the circle.

Note 4. The power of the point (x_1, y_1) with respect to the circle

$$Ax^2 + Ay^2 + 2Gx + 2Fy + C = 0$$

is

$$\frac{1}{A}(Ax_1^2 + Ay_1^2 + 2Gx_1 + 2Fy_1 + C).$$

Note 5. The expression for $PA \,.\, PB$ can also be obtained by using the relation

$$PA \,.\, PB = \pm(PC^2 - r^2)$$

where r is the radius of the circle and C is its centre.

Example. Show that the powers of the point (1, 2) *with respect to the circles*

$$x^2+y^2+4x-6y-3=0, \quad\text{.....................(i)}$$

and

$$2x^2+2y^2-3x-7y-5=0 \quad\text{.....................(ii)}$$

are equal.

Power with respect to (i) $=1+4+4-12-3=-6$.

" " " (ii) $=\frac{1}{2}(2+8-3-14-5)=-6$.

§ **61**. *Definition of Radical Axis.*

The radical axis of two circles is the locus of points whose powers with respect to the circles are equal.

§ **62.** *The radical axis of the circles*

$$x^2+y^2+2gx+2fy+c=0, \quad\text{.................(i)}$$

and

$$x^2+y^2+2g_1x+2f_1y+c_1=0, \quad\text{..............(ii)}$$

is the straight line

$$2(g-g_1)x+2(f-f_1)y+c-c_1=0.$$

Let $P(x_1, y_1)$ be any point on the radical axis of (i) and (ii); then

$$x_1{}^2+y_1{}^2+2gx_1+2fy_1+c=x_1{}^2+y_1{}^2+2g_1x_1+2f_1y_1+c_1,$$

i.e.

$$2(g-g_1)x_1+2(f-f_1)y_1+c-c_1=0,$$

i.e. P lies on the straight line

$$2(g-g_1)x+2(f-f_1)y+c-c_1=0,$$

which is therefore the radical axis.

Note 1. The gradient of the radical axis is $-\dfrac{g-g_1}{f-f_1}$, and the gradient of the line of centres is $\dfrac{f-f_1}{g-g_1}$; therefore the radical axis is perpendicular to the line of centres.

Note 2. If the circles (i) and (ii) intersect, then at the points of intersection

$$x^2+y^2+2gx+2fy+c=x^2+y^2+2g_1x+2f_1y+c_1,$$

i.e.

$$2(g-g_1)x+2(f-f_1)y+c-c_1=0. \quad\text{...................(iii)}$$

This is the equation of a straight line, satisfied by the coordinates of the points of intersection of the circles and therefore represents the common chord; the common chord is the radical axis in this case.

Note 3. If the circles do not intersect in real points, equation (iii) may be considered as the common chord, for the equation is satisfied by the imaginary coordinates which satisfy both (i) and (ii). The radical axis of two circles may therefore be defined as the line passing through the points common to the circles.

Note 4. The radical axes of three circles taken in pairs are concurrent, for, if two of the radical axes intersect at P, the powers of P with respect to all three circles are equal, and therefore P lies on the third radical axis. P is called the radical centre of the circles. If the centres of the three circles are collinear, the three radical axes are parallel and the radical centre is at infinity.

Example. Show that the difference of the powers of any point with respect to two circles varies as the distance of the point from their radical axis.

Let the circles be

$$x^2+y^2+2gx+2fy+c=0, \qquad \text{(i)}$$

and

$$x^2+y^2+2g_1x+2f_1y+c_1=0, \qquad \text{(ii)}$$

and let the point be $P(x_1, y_1)$.

The difference of the powers

$$=x_1^2+y_1^2+2gx_1+2fy_1+c-(x_1^2+y_1^2+2g_1x_1+2f_1y_1+c_1)$$

$$=2(g-g_1)x_1+2(f-f_1)y_1+c-c_1,$$

which varies as the distance of the point P from the line

$$2(g-g_1)x+2(f-f_1)y+c-c_1=0,$$

which is the radical axis of (i) and (ii).

§ **63.** *Definition of Coaxal Circles.*

A system of circles every two of which have the same radical axis is called a coaxal system.

Note. Since the radical axis of two circles is perpendicular to the line of centres, the centres of the circles of a coaxal system must be collinear.

§ **64.** *The equation*

$$x^2+y^2+2gx+c=0, \qquad \text{(i)}$$

where g can be given any value, represents a system of coaxal circles.

Consider any two circles of the system, say

$$x^2+y^2+2g_1x+c=0 \quad \text{.......................(ii)}$$

and

$$x^2+y^2+2g_2x+c=0. \quad \text{....................(iii)}$$

The radical axis of (ii) and (iii) is the line $x=0$, i.e. the y-axis; therefore the equation (i) represents a coaxal system, the common radical axis being the y-axis.

Note. The x-axis is the line of centres.

§ **65.** *To determine the conditions that the equation*

$$x^2+y^2+2gx+c=0 \quad \text{........................(i)}$$

should represent a (a) tangential, (b) intersecting, (c) non-intersecting system of circles.

The circles (i) meet the y-axis where

$$y^2+c=0;$$

$\therefore$ if $c=0$, the circles (i) touch the y-axis at the origin,
if c is $-$ive, the circles (i) cut the y-axis at the points $(0, \pm\sqrt{-c})$,
and if c is $+$ive, the circles (i) do not cut the y-axis at real points.

Therefore the equation (i) represents a tangential, intersecting or non-intersecting system of circles according as c is zero, negative or positive.

Note. The equation

$$x^2+y^2+2gx+c=0 \quad \text{..........................(i)}$$

represents a circle of radius $\sqrt{g^2-c}$; therefore if c is positive and $g=\pm\sqrt{c}$ we have two circles,

$$x^2+y^2\pm2\sqrt{c}x+c=0$$

of zero radius. These point circles at $(\mp\sqrt{c}, 0)$ are called the limiting points of the non-intersecting system represented by (i).

Example. Show that the length of the tangent from any point on the radical axis to any circle of a non-intersecting coaxal system is equal to the distance of the point from either limiting point.

Let the system have equation

$$x^2 + y^2 + 2gx + c = 0 \; ; \quad \text{.......................(i)}$$

then the limiting points are $(\mp\sqrt{c}, 0)$, and the radical axis is the y-axis.

Take any point $P(0, y_1)$ on the radical axis.

The distance of P from either limiting point

$$= \sqrt{y_1^2 + c}$$

= the length of the tangents from P to (i).

§ **66.** *If two circles have equations*

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad \text{...................(i)}$$

and $$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0, \quad \text{..................(ii)}$$

the equation

$$x^2 + y^2 + 2gx + 2fy + c + k(x^2 + y^2 + 2g_1x + 2f_1y + c_1) = 0, \text{ ..(iii)}$$

where k can be given any value, represents a system of circles coaxal with (i) *and* (ii).

For any given value of k, equation (iii) represents a circle, for the coefficients of x^2 and y^2 are equal and there is no term in xy. Also, equation (iii) is satisfied by the values of x and y which satisfy both (i) and (ii), and therefore represents a circle passing through the points of intersection, real or imaginary, of (i) and (ii), i.e. represents a circle coaxal with (i) and (ii). Therefore, when k can be given any value, (iii) represents a system of circles coaxal with (i) and (ii).

Example. Find the equation of the circle coaxal with the circles

$$x^2 + y^2 - 3x + 4y - 1 = 0$$

and $$2x^2 + 2y^2 + 5x - 6y + 3 = 0,$$

and passing through the point (1, 2).

The equation is

$$x^2 + y^2 - 3x + 4y - 1 + k(2x^2 + 2y^2 + 5x - 6y + 3) = 0,$$

where $$1+4-3+8-1+k(2+8+5-12+3)=0,$$
i.e. where $$k=-\tfrac{3}{2};$$
$\therefore$ the equation is
$$2(x^2+y^2-3x+4y-1)-3(2x^2+2y^2+5x-6y+3)=0,$$
i.e. $$4x^2+4y^2+21x-26y+11=0.$$

§ 67. *The equation*
$$x^2+y^2+2gx+2fy+c+k(Ax+By+C)=0, \quad \text{..........(i)}$$
where k can be given any value, represents a system of coaxal circles, the radical axis being the line
$$Ax+By+C=0. \quad \text{.......................(ii)}$$

For any given value of k, equation (i) represents a circle, for the coefficients of x^2 and y^2 are equal and there is no term in xy. Also, equation (i) is satisfied by the values of x and y which satisfy both (ii) and the equation
$$x^2+y^2+2gx+2fy+c=0, \quad \text{....................(iii)}$$
and therefore represents a circle passing through the points of intersection, real or imaginary, of (ii) and (iii). Therefore, when k can be given any value, (i) represents a system of coaxal circles, the radical axis being (ii).

Example. The circle
$$x^2+y^2+4x-6y+3=0$$
is one of a coaxal system having as radical axis the line
$$2x-4y+1=0;$$
find the circle of the system which touches the line
$$x+3y-2=0. \quad \text{............................(i)}$$

The circle has equation of the form
$$x^2+y^2+4x-6y+3+k(2x-4y+1)=0. \quad \text{..........(ii)}$$
Where (ii) meets (i),
$$(2-3y)^2+y^2+4(2-3y)-6y+3+k(2\,\overline{2-3y}-4y+1)=0,$$
i.e. $$10y^2-10(3+k)y+5(3+k)=0,$$
i.e. $$2y^2-2(3+k)y+(3+k)=0.$$

$\therefore$ (i) is a tangent to (ii) if

$$(3+k)^2 - 2(3+k) = 0,$$

i.e. if $$(3+k)(1+k) = 0,$$

i.e. if $$k = -1 \text{ or } -3.$$

Substituting these values in (ii), we get the equations

$$x^2 + y^2 + 2x - 2y + 2 = 0 \quad \text{.....................(iii)}$$

and $$x^2 + y^2 - 2x + 6y = 0. \quad \text{.....................(iv)}$$

(iii) represents a point circle and therefore the required equation is (iv).

§ **68.** *Definition of Orthogonal Circles.*

Two circles are said to cut orthogonally, or to be orthogonal, when the tangents to the circles at a point of intersection are perpendicular to each other.

Note. If the tangents at one point of intersection are perpendicular, those at the other must also be perpendicular.

§ **69.** *If the circles*

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{.....................(i)}$$

and $$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \text{.....................(ii)}$$

are orthogonal, $$2gg_1 + 2ff_1 = c + c_1.$$

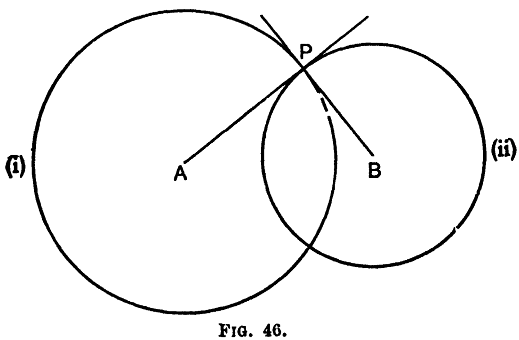

FIG. 46.

Let A, B (Fig. 46) be the centres of circles (i) and (ii) intersecting at P.

If the circles are orthogonal, AP and BP are tangents to (ii) and (i) respectively;

$$\therefore \quad A\hat{P}B = \text{right angle};$$

$$\therefore \quad AB^2 = AP^2 + PB^2;$$

$$\therefore \quad (g-g_1)^2 + (f-f_1)^2 = g^2 + f^2 - c + g_1{}^2 + f_1{}^2 - c_1,$$

i.e.
$$2gg_1 + 2ff_1 = c + c_1.$$

Example. Show that any circle passing through the limiting points of a coaxal system is orthogonal to every circle of the system.

Let the system have equation

$$x^2 + y^2 + 2gx + c = 0; \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

then the limiting points are $(\mp\sqrt{c}, 0)$.

Any circle through these points has equation of the form

$$x^2 + y^2 + 2fy - c = 0, \quad \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

and (i) and (ii) satisfy the condition for orthogonal circles.

Exercises

1. Show that the point $(2, -3)$ has equal powers with respect to the circles

$$x^2 + y^2 - 8x - 9y + 12 = 0$$

and
$$2x^2 + 2y^2 + x - 16y - 4 = 0.$$

2. Show that the point $(-1, 2)$ has equal powers with respect to the circles

$$x^2 + y^2 + 4x - 6y + 10 = 0$$

and
$$3x^2 + 3y^2 + 5x + 8y - 29 = 0.$$

3. Show that the powers of the point $(-2, 1)$ with respect to the circles

$$x^2 + y^2 + 4y - 19 = 0$$

and
$$3x^2 + 3y^2 - 14x - 12y - 1 = 0.$$

are equal in magnitude and opposite in sign.

4. Find the powers of the point $(-2, 1)$ with respect to the circles

$$x^2 + y^2 + 4x - 1 = 0$$

and
$$2x^2 + 2y^2 - 2x - 3y = 0;$$

show that the given point lies within the first circle and without the second.

5. Determine whether the point (2, 3) lies inside, on, or outside the circles

$$x^2+y^2+2x-4y=0,$$
$$x^2+y^2-2x-2y-8=0,$$
$$x^2+y^2-5x-3y+6=0.$$

6. Find the length of the tangents from the point $(-3, 2)$ to the circle

$$x^2+y^2-5x-2y+1=0.$$

7. A point P is such that its power with respect to the circle

$$x^2+y^2+4x-3y-1=0$$

exceeds its power with respect to the circle

$$2x^2+2y^2-3x+5y-4=0$$

by 12; show that P lies on the line

$$x-y-2=0.$$

8. A variable point P is such that its power with respect to the circle

$$x^2+y^2-4x+4y+3=0$$

is twice its power with respect to the circle

$$2x^2+2y^2+x-y-3=0;$$

show that the locus of P is a circle and find the coordinates of its centre and the length of its radius.

9. The tangents from a point P to the circle

$$x^2+y^2+4x-2y=0$$

are inclined to one another at the angle whose tangent is $\frac{4}{3}$, and P lies on the line

$$2x+y+1=0;$$

find the coordinates of P.

10. Find the radical axes of the following pairs of circles:

(i) $x^2+y^2-2y-4=0, \quad x^2+y^2-x+y-12=0,$

(ii) $x^2+y^2-x-y-2=0, \quad 3x^2+3y^2-4x-12=0,$

(iii) $x^2+y^2+3x+3y+2=0, \quad 2x^2+2y^2-6x-3y+5=0;$

show that circles (i) intersect, circles (ii) touch and circles (iii) do not meet.

11. Find the length of the common chord of the circles

$$x^2+y^2-2y-9=0$$

and

$$x^2+y^2+3x+y-6=0.$$

12. Find the radical centre of the circles

$$x^2+y^2+x-y-2=0, \quad x^2+y^2-3x-3y+2=0,$$

and $$2x^2+2y^2-9x-5y+7=0,$$

and show that it lies on each of the circles.

13. Show that the circles

$$x^2+y^2=5, \quad x^2+y^2-4x-1=0$$

and $$x^2+y^2-4x-2y+3=0$$

have a common point.

14. Show that the circles

$$x^2+y^2+4x+8y-9=0, \quad x^2+y^2-6x+y+7=0$$

and $$2x^2+2y^2-3x+6y-5=0$$

are concurrent.

15. Find the equation of the circle passing through the points $(-4, 2)$, $(-3, -1)$ and having its centre on the line

$$3x-y-1=0\,;$$

show that the chord common to this circle and the circle

$$x^2+y^2-2x+4y-4=0$$

is a diameter of the latter.

16. Show that one of the circles

$$x^2+y^2+2x-6y+2=0, \quad x^2+y^2+3x-5y+6=0$$

lies wholly within the other.

17. Prove that the circles

$$x^2+y^2-6x-16=0, \quad x^2+y^2-3y-19=0$$

and $$2x^2+2y^2-14x+y-31=0$$

are coaxal, and state the equation of the radical axis.

18. Find the coordinates of the centres and the lengths of the radii of the circles

$$x^2+y^2+2gx=0$$

when $g=\pm 1, \ \pm 2, \ \pm 3$; draw the circles.

19. Find the coordinates of the centres and the lengths of the radii of the circles

$$x^2+y^2+2gx+c=0$$

when $g=0, \ \pm 1, \ \pm 2$ and $c=-4$; draw the circles.

20. Find the coordinates of the centres and the lengths of the radii of the circles

$$x^2+y^2+2gx+c=0$$

when $g=\pm 2, \ \pm 2\frac{1}{2}, \ \pm 3, \ \pm 4$ and $c=4$; draw the circles.

21. Find the equation of the circle which passes through the point $(-3, 2)$ and which is coaxal with the circles

$$3x^2+3y^2+2x-7y-6=0, \quad x^2+y^2-y+2=0.$$

22. Find the equation of the circle which has as diameter the chord common to the circles

$$x^2+y^2-2x+2y-3=0, \quad 5x^2+5y^2-x+7y-12=0.$$

23. Show that the line

$$4x-2y+3=0$$

touches the circle

$$2x^2+2y^2-6x+2y-5=0,$$

and determine the point of contact; find also the equation of the circle which touches the given line at the same point and which passes through the point $(1, 2)$.

24. Show that the equation

$$x(x-1)+y(y-2)=k(2x+y-2)$$

represents for all values of k a circle passing through the points $A(1, 0)$ and $B(0, 2)$; hence determine the equations of the circles passing through A and B and touching the line, $x=5$.

25. Two circles S_1, S_2 pass through the points common to the circles

$$x^2+y^2-2x-16=0, \quad x^2+y^2+4x-6y+8=0;$$

S_1 passes through the point $(3, 1)$ and S_2 passes through the centre of S_1; find the equations of S_1 and S_2, and show that S_2 has as diameter the common chord of the given circles.

26. Show that, if P is any point on the radical axis of two circles, the polars of P with respect to the circles meet on the radical axis.

27. Show that the polar of a fixed point P with respect to any circle of a coaxal system passes through a fixed point Q and that the radical axis bisects PQ.

28. Prove that the circles

$$2x^2+2y^2-3x-4y+2=0, \quad x^2+y^2-4x+2y=0$$

intersect orthogonally.

29. A, B, C, D, E are the points $(-4, 3)$, $(2, -5)$, $(6, 2)$, $(6, -2)$, $(2, 0)$; find the equations of (i) the circle on AB as diameter, (ii) the circle CDE, and prove that these circles are orthogonal.

30. Find the equation of the circle which touches the line

$$3x-y-6=0$$

at the point $(2, 0)$ and passes through the point $(0, -2)$, and of the circle which has the points $(-2, 5)$, $(1, 4)$ at the extremities of a diameter; prove that the circles are orthogonal.

31. A circle passes through the point (h, k) and cuts orthogonally the circle
$$x^2+y^2+2gx+2fy+c=0\,;$$
show that its centre lies on the line
$$2(h+g)x+2(k+f)y=h^2+k^2-c.$$

32. Prove analytically that, if a circle S cuts orthogonally the circles S_1 and S_2, the centre of S lies on the radical axis of S_1 and S_2.

33. Find the equation of the circle which is orthogonal to the circles
$$x^2+y^2+2x-2y+1=0, \quad x^2+y^2+4x-4y+3=0$$
and whose centre lies on the line
$$3x-y-2=0.$$

34. Find the equation of the circle which is orthogonal to the circles
$$x^2+y^2+3x-5y+6=0, \quad 4x^2+4y^2-28x+29=0$$
and whose centre lies on the line
$$3x+4y+1=0.$$

35. P is a variable point on the line
$$2x+y-1=0,$$
and Q is its inverse with respect to the circle
$$x^2+y^2=4\,;$$
show that the locus of Q has equation
$$x^2+y^2-8x-4y=0,$$
and determine the equation of the circle which is orthogonal to this and the given circle and whose centre lies on the line
$$x+y+1=0.$$

36. Find the equation of the circle which cuts orthogonally the circles
$$x^2+y^2=5, \quad x^2+y^2+6x+1=0,$$
$$x^2+y^2-4x-4y+7=0.$$

37. Show that the circle which is orthogonal to the circles
$$x^2+y^2+4x+6y-5=0, \quad x^2+y^2+8x+y-20=0$$
and
$$x^2+y^2+6x+2y-14=0$$
is orthogonal to the circle
$$x^2+y^2-6x+16y+30=0.$$

38. Find the equation of the circle which is orthogonal to the circles
$$x^2+y^2+3x-6y+5=0, \quad x^2+y^2-7x-y=0$$
and which passes through the point $(-3, 0)$.

39. Show that each circle of the system

$$x^2+y^2+ky-c=0$$

is orthogonal to each circle of the system

$$x^2+y^2+k_1x+c=0\ ;$$

hence determine the equations of the circles which are orthogonal to the circle

$$x^2+y^2+10x+1=0$$

and which touch the line

$$3x-y-7=0.$$

40. Find the equation of the circle passing through the point (3, 1) and orthogonal to the circles

$$x^2+y^2-8x+4=0, \quad x^2+y^2-6x+4=0.$$

REVISION EXERCISES

D. On Chapters IX-X

1. Find the centres and length of the radii of the circles

(i) $x^2+y^2-2x+4y+1=0$, (ii) $x(x-3)+y(y+2)=0$,

(iii) $2x^2+2y^2-6x-10y-1=0$,

(iv) $x^2+y^2-a^2$

$$=2a\cos\frac{\theta-\varphi}{2}\left(x\cos\frac{\theta+\varphi}{2}+y\sin\frac{\theta+\varphi}{2}-a\cos\frac{\theta-\varphi}{2}\right).$$

2. The circle

$$x^2+y^2-4x-4y-5=0$$

meets OX, OY in A_1, B_1 and OX', OY' in A_2, B_2 respectively; show that

$$\text{(i) } A_1B_1=5A_2B_2, \quad \text{(ii) } A_1B_1 \parallel A_2B_2.$$

3. Show that the circles

$$x^2+y^2-2x-6y-15=0, \quad x^2+y^2-6x+2y-7=0$$

make equal intercepts on the x-axis.

4. The circles

$$x^2+y^2+2gx+2fy+c=0, \quad x^2+y^2+2g_1x+2f_1y+c_1=0$$

make equal intercepts on the x-axis; show that

$$g^2-g_1{}^2=c-c_1.$$

5. Find the length of the chord

$$x-y+1=0$$

of the circle

$$x^2+y^2-2x+2y-3=0.$$

6. Find the length of the chord

$$x-2y+1=0$$

of the circle

$$4x^2+4y^2-18x+12y-47=0.$$

7. Find the mid-point of the common chord of the circles

$$x^2+y^2-4y-6=0, \quad x^2+y^2+4x-2y-20=0.$$

8. Find the length of the common chord of the circles

$$x^2+y^2+2x-11y=0, \quad x^2+y^2-10x+5y=0.$$

9. Show that the triangles with vertices

$$\text{(i) } (4, 1), (3, 4), (-4, -3),$$
$$\text{(ii) } (-1, -4), (2, -3), (3, -2)$$

have the same circumcircle.

10. Show that the points $(0, 0)$, $(4, 2)$, $(3, 4)$, $(1\frac{1}{2}, 4\frac{1}{2})$ lie on a circle, and find the intercepts which the circle makes on the x- and y-axes.

11. The line

$$2x-y+6=0$$

meets the circle

$$x^2+y^2-2y-9=0$$

at A, B; find the equation of the circle on AB as diameter.

12. The line

$$x+y=a$$

is a chord of the circle

$$x^2+y^2+4x-4y+8=a^2;$$

show that the circle having this chord as diameter passes through the centre of the given circle.

13. Show that the equation of the circle which has as diameter the chord common to the circles

$$x^2+y^2-ax=0, \quad x^2+y^2-by=0$$

is

$$(a^2+b^2)(x^2+y^2)=ab(bx+ay).$$

14. Determine the equation of the circle which has as diameter the line joining the origin to the point (a, b) and find the lengths of the intercepts OA, OB which the circle makes on the x- and y-axes; find the equations of the circles on OA and OB as diameters and show that they intersect on the line

$$ax=by.$$

15. Interpret the polar equations

$$\text{(i) } r=2a\cos\theta, \quad \text{(ii) } r=2a\cos(\theta-\alpha).$$

16. The equation

$$y=mx$$

represents, for different values of m, a system of chords of the circle

$$x^2+y^2-4x-8y=0;$$

show that the mid-points of the chords lie on the circle

$$x^2+y^2-2x-4y=0.$$

17. The line
$$x-y-1=0$$
is a chord of the circle
$$x^2+y^2-14x+2y+21=0;$$
find the ratio in which the chord is divided by the circle
$$x^2+y^2-10x-2y+21=0.$$

18. Show that the chord of the circle
$$x^2+y^2=a^2,$$
having the point (x_1, y_1) as its mid-point, has equation
$$xx_1+yy_1=x_1{}^2+y_1{}^2.$$

19. Through the point (3, 4) on the circle
$$4x^2+4y^2-24x-7y=0$$
two chords of length 5 are drawn; find their equations.

20. The circle $$x^2+y^2=r^2$$
has a chord whose equation is
$$\frac{x}{a}+\frac{y}{b}=1;$$
show that the chord is equal in length to the radius if
$$3r^2(a^2+b^2)=4a^2b^2.$$

21. The line passing through the centres of the circles
$$x^2+y^2+2gx+2fy=0, \quad x^2+y^2+2g_1x+2f_1y=0$$
meets the x- and y-axes in A, B respectively; shew that the circle on AB as diameter has equation
$$(g-g_1)(f-f_1)(x^2+y^2)-(gf_1-g_1f)(\overline{g-g_1}x-\overline{f-f_1}y)=0.$$

22. Show that the equation
$$(ax+by+c)^2+(bx-ay+d)^2=e$$
represents a circle of which the lines
$$ax+by+c=0, \quad bx-ay+d=0$$
are perpendicular diameters.

23. AB is a diameter of the circle
$$x^2+y^2-4x-5=0$$
and CD, a diameter of the circle
$$x^2+y^2-6y+5=0,$$
is perpendicular to AB; show that D is the orthocentre of $\triangle ABC$.

24. Show that the tangent at the point (x_1, y_1) on the circle
$$(x-h)^2+(y-k)^2=a^2$$
has equation $$(x-h)(x_1-h)+(y-k)(y_1-k)=a^2.$$

25. Show that the line
$$(x - a\cos\theta)(x_1 - a\cos\theta) + (y - a\sin\theta)(y_1 - a\sin\theta) = a^2$$
touches the circle
$$x^2 + y^2 - 2ax\cos\theta - 2ay\sin\theta = 0,$$
when (x_1, y_1) is a point on the circle.

26. The line
$$3x + ky - 1 = 0$$
touches the circle
$$x^2 + y^2 - 8x - 2y + 4 = 0\,;$$
find k.

27. Show that the tangent at the point $(2, -1)$ to the circle
$$x^2 + y^2 - 6x - 2y + 5 = 0$$
touches the circle
$$x^2 + y^2 + 6x + 22y + 5 = 0.$$

28. Show that the tangents to the circle
$$x^2 + y^2 + 6x + 2y - 3 = 0$$
at the points $(0, 1)$, $(-1, -4)$ are inclined at 45° to the chord of contact.

29. Show that the lines
$$2x^2 + 5xy + 2y^2 - 7x - 14y = 0$$
are tangents to the circle
$$x^2 + y^2 - 6x + 8y + 20 = 0.$$

30. The line
$$y = mx$$
touches the circle
$$x^2 + y^2 + 2gx + 2fy + c = 0\,;$$
prove that
$$(c - f^2)m^2 - 2fgm + (c - g^2) = 0.$$

31. AB and CD, chords of the circle
$$x^2 + y^2 = 5,$$
have equations
$$x + y + 1 = 0, \quad x - y - 1 = 0\,;$$
the tangents at A and B meet at P, and the tangents at C and D meet at Q; show that PQ is the line
$$y + 5 = 0.$$

32. Find the equation of the diagonals of the parallelogram which has as sides the tangents to the circle
$$x^2 + y^2 = 5$$
at the points of intersection of the circle and the lines
$$2x^2 - 5xy + 2y^2 = 0.$$

33. The tangent at the point (x_1, y_1) to the circle

$$x^2+y^2+2gx+2fy+c=0$$

touches the circle

$$x^2+y^2=a^2;$$

prove that

$$(gx_1+fy_1+c)^2=a^2(g^2+f^2-c).$$

34. Find the equations of the chords of the circle

$$x^2+y^2-2x-4y-20=0,$$

which have length $4\sqrt{5}$ and gradient $\frac{1}{2}$.

35. The point $(4, -1)$ is the mid-point of a chord of the circle

$$x^2+y^2-6x-2y-15=0;$$

find the length of the chord and show that the tangents at its extremities meet at the point $(8, -9)$.

36. Find the equations of the tangents to the circle

$$x^2+y^2-8x+2y-8=0,$$

which are perpendicular to the line

$$3x+4y+1=0,$$

and show that the diameter joining the points of contact touches the circle

$$x^2+y^2-2x+10y+1=0.$$

37. Find the equation of the circle passing through the points $A(-3, 1)$, $B(1, -5)$ and $C(2, 0)$. The tangents at A and B make an intercept DE on the tangent at C; find the length of DE and show that DE subtends a right-angle at the centre of the circle.

38. Find the equations of the tangents to the circles

$$x^2+y^2-4x-2y=0, \quad x^2+y^2-8x-4y+10=0$$

at their points of intersection.

39. Show that the circle

$$2x^2+2y^2+2x-5y-2=0$$

touches the circle

$$x^2+y^2-4x-5y+9=0,$$

and find the equation of the tangent at the point of contact.

40. Show that each of the circles

$$x^2+y^2+4y-1=0, \quad x^2+y^2+6x+y+8=0,$$

$$x^2+y^2-4x-4y-37=0$$

touches the other two; determine the points of contact and the point of concurrence of the tangents at these points.

41. Show that the circles

$$x^2+y^2+2gx+2fy+c=0, \quad x^2+y^2=a^2$$

touch each other if

$$(a^2+c)^2=4a^2(g^2+f^2).$$

42. Prove that the circles

$$x^2+y^2+4x-6y-12=0, \quad x^2+y^2-3y-4=0$$

have internal contact.

43. Prove that the circles

$$2x^2+2y^2+3x-y-1=0, \quad 2x^2+2y^2-5x+7y+3=0$$

have external contact.

44. Prove that the circles

$$(x-h)^2+(y-k)^2=a^2, \quad (x-h_1)^2+(y-k_1)^2=a_1^2$$

touch externally or internally according as

$$(h-h_1)^2+(k-k_1)^2=(a\pm a_1)^2.$$

45. Prove that the circles

$$x^2+y^2+2gx+2fy+c=0, \quad x^2+y^2+2g_1x+2f_1y+c_1=0$$

touch externally or internally according as

$$2(gg_1+ff_1\pm rr_1)=c+c_1$$

when r, r_1 are the radii.

46. Find the equation of the circle passing through the point (3, 2) and touching the line

$$4x-3y-11=0$$

at the point (2, -1).

47. Find the equation of the circle of minimum radius, passing through the point (1, 3) and touching the circle

$$2x^2+2y^2-9x-2y+5=0.$$

48. Find the equations of the circles which pass through the points (-1, 4), (1, 2) and which touch the line

$$3x-y-3=0.$$

49. Find the equation of the circle which passes through the points (0, 2), (1, 5) and has its centre on the line

$$x+5y-15=0;$$

show that the point (2, 6) lies on the circle, find the equation of the tangent at this point, show that the tangent touches the circle

$$4x^2+4y^2-12x-20y+9=0$$

and determine the point of contact with this circle.

50. A circle touches the y-axis and makes intercept a on the x-axis; show that its centre lies on the curve

$$4(x^2-y^2)=a^2.$$

51. A circle passes through the point (3, 1) and touches the line

$$x-3y-5=0;$$

show that its centre lies on the curve

$$(3x+y)^2=25(2x+2y-3).$$

52. A circle touches the line

$$y=2x$$

and makes intercept $2a$ on the y-axis; show that the locus of its centre has equation

$$x^2+4xy-y^2+5a^2=0.$$

53. Find the equations of the circles which pass through the point (4, 3), touch the y-axis and have their centres on the line

$$3x=2y.$$

54. Find the equation of the circle passing through the points $A(-5, 0)$, $B(1, 0)$, $C(2, 1)$, and show that the line

$$4x-3y-5=0$$

is a tangent to it; determine the equation of the other circle which touches this line and which passes through A and B.

55. Find the equations of the circles touching the line

$$y=2$$

at the point (3, 2) and making an intercept of 8 on the y-axis.

56. Find the equations of the circles which pass through the point $(-1, 5)$ and touch the lines

$$x-2y+13=0, \quad 2x+y+6=0.$$

57. Show that the line

$$x-2y+4=0$$

is a common tangent to the circles

$$5x^2+5y^2=16, \quad x^2+y^2+2x+2y-3=0;$$

find the other common tangent.

58. Find the equations of the common tangents to the circles

$$x^2+y^2-2y-4=0, \quad x^2+y^2-6x-11=0.$$

59. Find the equation of the pair of transverse common tangents to the circles

$$x^2+y^2=5, \quad x^2+y^2-24x-12y+160=0.$$

60. Find the equations of the four common tangents to the circles

$$5x^2+5y^2+10x+10y+6=0, \quad 5x^2+5y^2+30x+30y+74=0.$$

61. Common tangents are drawn to the circles

$$(x-h)^2+(y-k)^2=a^2 \quad \text{..........................(i)}$$

and

$$(x-h_1)^2+(y-k_1)^2=a_1^2 ;$$

show that the chord of (i) joining the points of contact of direct tangents has equation

$$(h_1-h)(x-h)+(k_1-k)(y-k)=a(a-a_1)$$

and that the chord joining the points of contact of transverse tangents is

$$(h_1-h)(x-h)+(k_1-k)(y-k)=a(a+a_1).$$

62. Find the equations of the pairs of tangents from the origin to the following circles :

(i) $x^2+y^2+4x-2y+4=0,$

(ii) $(x-1)(x-3)+(y-1)(y-7)=0,$

(iii) $3x^2+3y^2-5x+2y+1=0.$

63. Find the equations of the tangents from the following points to the given circles :

(i) $(3, -1),\quad x^2+y^2=5 ;$

(ii) $(5, -1),\quad x^2+y^2=13 ;$

(iii) $(-3, -6),\quad x^2+y^2+6y=0 ;$

(iv) $(1, -4),\quad x^2+y^2+4x+6y+8=0 :$

(v) $(3, 0),\quad x^2+y^2+2x-4y+1=0.$

64. Find the equation of the line which bisects the tangents from the point $(-1, -10)$ to the circle

$$x^2+y^2-8x-10y+16=0.$$

65. Show that the line

$$x-2y+5=0$$

passes through the point $P(-1, 2)$ and touches the circle

$$x^2+y^2-4x-2y=0 ;$$

determine the equation of the other tangent from P.

66. Show that the points $P(a \cos \theta, a \sin \theta)$, $Q(a \cos \varphi, a \sin \varphi)$ lie on the circle

$$x^2+y^2=a^2,$$

and that the equation of the chord PQ is

$$x \cos \frac{\theta+\varphi}{2}+y \sin \frac{\theta+\varphi}{2}=a \cos \frac{\theta-\varphi}{2} ;$$

deduce that the tangent at the point P has equation

$$x \cos \theta+y \sin \theta=a.$$

67. Chords of the circle

$$x^2+y^2+2gx+2fy+c=0$$

pass through the point (x_1, y_1); prove that the mid-points of the chords lie on the circle

$$(x-x_1)(x+g)+(y-y_1)(y+f)=0.$$

E. On Chapters XI-XII

68. Find the polars of the points $(1, 2)$, $(4, -1)$, $(\frac{2}{3}, -\frac{1}{3})$ with respect to the circle

$$x^2+y^2-x+y=0,$$

and show that the polar of each point passes through the other two given points.

69. Show that, for all values of k, the points $(1, -2)$, $(3, 2)$ are conjugate with respect to the circle

$$3x^2+3y^2+2x+ky-1=0.$$

70. The points $(-1, 2)$, $(-2, 3)$ are conjugate with respect to a variable circle passing through the origin; show that the centre of the circle lies on the line

$$3x-5y+8=0.$$

71. Find the locus of the centre of a circle which passes through the point $(1, 2)$ and with respect to which the points $(-2, 1)$, $(3, -2)$ are conjugate.

72. The polar of each of the points $(6, -3)$, $(-3, -12)$, $(-1, -2)$ with respect to a circle passes through the other two points; find the equation of the circle.

73. The polars of the origin with respect to the circles

$$x^2+y^2+2gx+c=0, \quad x^2+y^2+2fy+c=0$$

meet at P; show that the line joining P to the origin passes through the points of intersection of the circles.

74. Find the inverse of the point $(1, -2)$ with respect to the circle

$$2x^2+2y^2=5.$$

75. Find the inverse of the point $(1, 1)$ with respect to the circle

$$x^2+y^2+6x+2y-15=0.$$

76. The inverse of $P(x, y)$ with respect to the circle

$$x^2+y^2=4$$

lies on the circle

$$x^2+y^2-3x+y=0;$$

show that

$$3x-y-4=0.$$

77. P is a variable point on the circle

$$x^2+y^2-2x+3y=0\ ;$$

show that the inverse of P with respect to the circle

$$x^2+y^2=2$$

has locus

$$2x-3y-2=0.$$

78. P lies on the circle

$$x^2+y^2+x-y=0\ ;$$

find the locus of the inverse of P with respect to the circle

$$x^2+y^2=8.$$

79. P lies on the line

$$3x-2y+1=0\ ;$$

show that the inverse of P with respect to the circle

$$x^2+y^2=3$$

lies on the circle

$$x^2+y^2+9x-6y=0.$$

80. P lies on the line

$$2x-3y+4=0\ ;$$

find the locus of the inverse of P with respect to the circle

$$x^2+y^2=1.$$

81. A point moves on the circle

$$x^2+y^2-5x-3y+6=0\ ;$$

show that its inverse with respect to the circle

$$x^2+y^2-2x-2y+1=0$$

has locus

$$3x+y-5=0.$$

82. Find the points whose powers with respect to the circles

$$x^2+y^2=3,$$
$$x^2+y^2-x=0,$$
$$x^2+y^2+3x+2y-6=0$$

are in the ratio 1 : 2 : 3.

83. Find the point whose powers with respect to the circles

$$x^2+y^2=5,$$
$$2x^2+2y^2-7x=0,$$
$$x^2+y^2+2x-5y+2=0,$$
$$3x^2+3y^2-11y=0$$

are as 4 : 3 : 2 : 1.

84. The power of P with respect to the circle

$$x^2+y^2=3$$

is three times its power with respect to the circle

$$x^2+y^2-2x-1=0;$$

show that the locus of P is a circle passing through the origin.

85. Find the equation of the circle which passes through the points $(1, 0)$, $(2, 1)$ the length of the tangent to which from the point $(4, -1)$ is $2\sqrt{5}$.

86. Find the equation of the circle which passes through the points $(1, 2)$, $(1, -1)$ with respect to which the power of the point $(2, 4)$ is 10.

87. Find the equation of the common chord of the circles

$$x^2+y^2-12x-6y-80=0, \quad x^2+y^2-6x+2y-40=0,$$

and show that the circle having the common chord as diameter touches the x-axis at the origin.

88. A, B, C are the points (a, b), (b, c), (c, a); show that the common chord of the circles on AC, BC as diameters has equation

$$x(a-b)+y(b-c)+b(c-a)=0.$$

89. Find the equation of the circle coaxal with the circles

$$x^2+y^2-5x-5y+10=0, \quad x^2+y^2+3x-9y+10=0$$

and passing through the point $(5, 4)$.

90. Find the equation of the circle passing through the point $(2, 4)$ and the points of intersection of the line

$$x-3y+6=0$$

and the circle

$$x^2+y^2-4x-2y=0.$$

91. Show that the circle coaxal with the circles

$$x^2+y^2+x-7y+10=0, \quad x^2+y^2-4x-2y-5=0$$

and passing through the origin touches the line

$$11x+7y-54=0;$$

determine the point of contact.

92. Show that the equation

$$x^2+y^2-2x-2ay-8=0$$

represents, for different values of a, a system of circles passing through two fixed points A, B on the x-axis, and find the equation of that circle of the system the tangents to which at A, B meet on the line

$$x+2y+5=0.$$

93. Find the equation of the circle passing through the origin and the points of contact of tangents from the origin to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

94. Find the power of the point (3, 2) with respect to the circle

$$2x^2+2y^2-3x+5y-2=0,$$

and hence find the equation of the circle which has centre (3, 2) and which cuts the given circle orthogonally.

95. Find the equation of the circle with centre $(-1, 3)$ and cutting orthogonally the circle

$$3x^2+3y^2-2x+4y-1=0.$$

96. Find the equation of the circle passing through the point (3, 2) and cutting orthogonally the circles

$$x^2+y^2-7x-3y+12=0, \quad x^2+y^2-x-6y+3=0.$$

97. The point which divides in the ratio $k:1$ the line joining the points (x_1, y_1) and (x, y) lies on the circle

$$F(x, y) \equiv x^2+y^2+2gx+2fy+c=0\ ;$$

show that

$$k^2F(x, y)+2k(xx_1+yy_1+g\overline{x+x_1}+f\overline{y+y_1}+c)+F(x_1, y_1)=0,$$

and deduce that the equation of

(i) the tangent at a point (x_1, y_1) is

$$xx_1+yy_1+g\overline{x+x_1}+f\overline{y+y_1}+c=0,$$

(ii) the tangents from a point (x_1, y_1) is

$$F(x, y)F(x_1, y_1)=(xx_1+yy_1+g\overline{x+x_1}+f\overline{y+y_1}+c)^2,$$

(iii) the polar of a point (x_1, y_1) is

$$xx_1+yy_1+g\overline{x+x_1}+f\overline{y+y_1}+c=0.$$

98. One of the lines

$$x^2-4xy+3y^2+4x-12=0$$

meets the circle

$$x^2+y^2-4x-2y=0$$

at A, B, the other at C, D; find the equation of the line pairs AC, BD and BC, AD.

99. The lines

$$L_1 \equiv A_1x+B_1y+C_1=0, \quad L_2 \equiv A_2x+B_2y+C_2=0$$

cut the circle

$$S \equiv x^2+y^2+2gx+2fy+c=0$$

at the points A, B and C, D respectively; show that the equation of the lines AC, BD is of the form

$$S-kL_1L_2=0,$$

where k is a constant.

100. The line

$$P \equiv Ax + By + C = 0$$

meets the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

at A, B; show that the equation of the tangents at A, B is of the form

$$S - kP^2 = 0,$$

where k is a constant, and deduce the equation of the tangents from the point (x_1, y_1) to the circle.

101. Prove that the lines

$$3x^2 + 2xy - y^2 + 5x + y + 2 = 0$$

intersect the lines

$$4x^2 + 5xy - 6y^2 - 3x + 5y - 1 = 0$$

in four points which lie on a circle, and find the equation of the circle.

102. The lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

cut the lines

$$a_1x^2 + 2h_1xy + b_1y^2 + 2g_1x + 2f_1y + c_1 = 0$$

in four concyclic points; show that

$$\frac{a-b}{h} = \frac{a_1 - b_1}{h_1}.$$

103. The sides, taken in order, of a cyclic quadrilateral have equations

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_3x + b_3y + c_3 = 0,$$

and

$$a_4x + b_4y + c_4 = 0;$$

show that

$$(a_1a_3 - b_1b_3)(a_2b_4 + a_4b_2) = (a_2a_4 - b_2b_4)(a_1b_3 + a_3b_1).$$

PART III.—CONIC SECTIONS

CHAPTER XIII

CONIC SECTIONS. STANDARD EQUATIONS

§ **70.** *Definition of Conic Sections.*

The locus of a point which moves in a plane so that its distance from a fixed point in the plane bears a constant ratio to its distance from a fixed straight line in the plane is called a conic section.

The fixed point is called the focus, the fixed line the directrix and the constant ratio the eccentricity of the curve.

A conic section is called a parabola, ellipse or hyperbola according as the eccentricity is equal to, less than or greater than unity.

Note. The intersection of a plane and a right circular cone may be a parabola, ellipse or hyperbola according to the position of the plane; hence these curves are called conic sections or, briefly, conics.

§ **71.** *A conic is represented by an equation of the second degree.*

If the focus is the point (x_1, y_1), the directrix the line

$$ax + by + c = 0$$

and the eccentricity e, the equation of the conic is

$$(x - x_1)^2 + (y - y_1)^2 = e^2 \frac{(ax + by + c)^2}{a^2 + b^2}.$$

which is an equation of the second degree.

Note. By suitable choice of focus and directrix, the equations of conics can be written in simple forms; these standard equations are established in §§ 72, 74, 76.

Example. Find the equation of the ellipse with focus (2, −1), *directrix* $x-y+1=0$ *and eccentricity* $\frac{1}{2}\sqrt{2}$.

The equation is

$$(x-2)^2+(y+1)^2=\frac{1}{2}\,\frac{(x-y+1)^2}{2},$$

i.e.
$$3x^2+2xy+3y^2-18x+10y+19=0.$$

Exercises

Find the equations of the parabolas with the following foci and directrices :

1. $(-1, 1)$, $x+y-1=0$.
2. $(2, 1)$, $2x+y+1=0$.
3. $(0, 0)$, $x-2y+2=0$.
4. $(1, -1)$, $x-y=0$.
5. $(1, 0)$, $x=0$.

Find the equations of the ellipses with the following foci, directrices and eccentricities :

6. $(1, 2)$, $x+y+1=0$, $\frac{1}{2}$.
7. $(0, -1)$, $x+2y-1=0$, $\frac{1}{3}\sqrt{3}$.
8. $(0, 0)$, $2x+2y+1=0$, $\frac{2}{3}$.
9. $(1, 1)$, $x+2y=0$, $\frac{1}{2}\sqrt{2}$.
10. $(-1, 0)$, $x+2=0$, $\frac{1}{2}$.

Find the equations of the hyperbolas with the following foci, directrices and eccentricities :

11. $(-1, 1)$, $x-y+1=0$, 2.
12. $(1, 2)$, $3x-4y+1=0$, $\frac{5}{2}$.
13. $(0, 0)$, $x+y-1=0$, $\frac{3}{2}$.
14. $(1, 1)$, $x+2y=0$, 4.
15. $(1, 0)$, $x+2=0$, 2.

§ 72. *Standard equation of the parabola.*

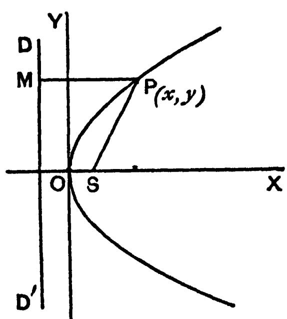

FIG. 47.

Let the focus S (Fig. 47) be the point $(a, 0)$, and let the directrix DD' be the line,

$$x + a = 0.$$

Let $P(x, y)$ be any point on the curve; draw PM perpendicular to DD'.

Then
$$PS = PM;$$
$$\therefore\ PS^2 = PM^2;$$
$$\therefore\ (x-a)^2 + y^2 = (x+a)^2,$$
i.e.
$$y^2 = 4ax.$$

This is the standard equation of the parabola.

§ **73.** *Form of the parabola, $y^2 = 4ax$, where a is positive.*

If the point (x, y) lies on the curve, $y^2 = 4ax$, the point $(x, -y)$ also lies on it; therefore the curve is symmetrical about the x-axis.

For real values of y, y^2 and therefore x must be positive but can have any value however great; therefore no part of the

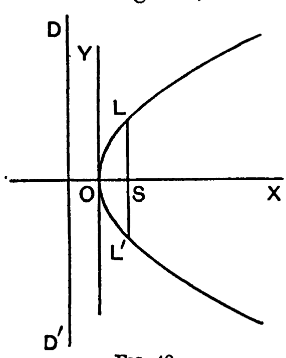

FIG. 48.

curve lies on the left of the y-axis, but the curve extends to infinity on the right of the y-axis.

The origin, midway between the focus and the directrix, is a point on the curve, and if in the equation, $y^2 = 4ax$, we make x

zero, we get $y^2=0$; therefore the y-axis meets the curve in two coincident points at the origin and is therefore the tangent at the origin.

When $x=a$, $y=\pm 2a$; therefore LL' (Fig. 48), the double ordinate through the focus, has length $4a$.

Note 1. In any parabola the line which passes through the focus and is perpendicular to the directrix is called the axis of the parabola, and the point of intersection of the curve and its axis is called the vertex. The chord which passes through the focus and is perpendicular to the axis is called the latus rectum. In the parabola, $y^2=4ax$, therefore, the axis is the x-axis, the vertex is the origin and the latus rectum has length $4a$, while the y-axis is the tangent at the vertex.

Note 2. Since a conic is represented by an equation of the second degree, any straight line will meet a conic in two points, real or imaginary. A line is a tangent to a conic if it meets the conic in two coincident points.

Note 3. The equation, $y^2=4ax$, where a is negative, represents a parabola whose axis is the x-axis and whose vertex is the origin, the curve lying wholly to the left of the y-axis.

Example 1. *Show that the equation,*

$$y^2-x+4y+5=0,$$

represents a parabola; find the axis, vertex and directrix.

The equation can be written,

$$(y+2)^2=x-1.$$

Changing the origin to the point $(1, -2)$, this equation becomes

$$y^2=x,$$

and therefore represents a parabola with the new x-axis as axis, the new origin as vertex and latus rectum of unit length.

Referred to the original axes, the parabola has axis,

$$y+2=0,$$

vertex $(1, -2)$ and directrix,

$$x = 1 - \tfrac{1}{4},$$

i.e.

$$4x = 3.$$

Example 2. Find the equation of the parabola with focus $(1, 1)$ *and vertex* $(-1, 1)$.

Distance between focus and directrix $= 2$;

$\therefore$ distance between vertex and directrix $= 2$;

$\therefore$ equation of directrix is $x = -3$;

$\therefore$ equation of parabola is

$$(x-1)^2 + (y-1)^2 = (x+3)^2,$$

i.e.

$$y^2 - 8x - 2y - 7 = 0.$$

Exercises

Find the lengths of the latera recta of the parabolas:

1. $y^2 = 8x$. **2.** $(y-2)^2 = 2(x+1)$.

3. $(y+2)^2 + 4(x+3) = 0$. **4.** $y^2 - 6x + 4y = 0$.

Find the equations of the parabolas with the following vertices and directrices:

5. $(1, 2)$, $x + 1 = 0$. **6.** $(2, 1)$, $x + 3 = 0$.

Find the equations of the parabolas with the following foci and vertices:

7. $(-2, -1)$, $(-1, -1)$. **8.** $(2, 1)$, $(1, 1)$.

9. $(3, 2)$, $(1, 2)$.

Find the vertices and directrices of the parabolas:

10. $y^2 - 4x - 4y = 0$. **11.** $y^2 + 8x - 4y - 4 = 0$.

Find the foci and directrices of the parabolas:

12. $y^2 + 2x - 2y - 4 = 0$. **13.** $y^2 - 4x + 2y - 3 = 0$.

14. $y^2 - 8x - 2y + 17 = 0$. **15.** $(y-k)^2 = 4a(x-h)$.

16. $x^2 - 6x - 4y + 17 = 0$

Sketch the parabolas:

17. $x^2 = 4y$. **18.** $x^2 = -4y$.

19. $(y-k)^2 = 4(x-h)$. **20.** $(x-h)^2 = 4(y-k)$.

§ **74.** *Standard equation of the ellipse.*

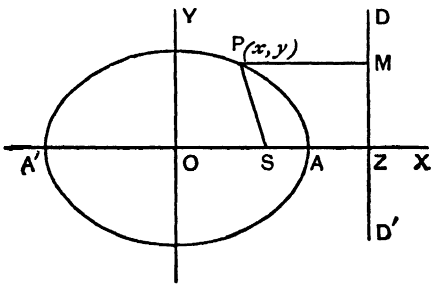

FIG. 49.

Let S (Fig. 49) be the focus and DD' the directrix. Draw SZ perpendicular to DD'. Let A, A' divide SZ internally and externally in the ratio $e:1$ where e is the eccentricity; then A, A' are points on the ellipse. Take O the mid-point of AA' as origin, OZ as x-axis and OY, perpendicular to OZ, as y-axis; then

$$A'S = eA'Z,$$

and
$$SA = eAZ;$$

$$\therefore \; A'A = e(A'Z + AZ) = 2e \,.\, OZ,$$

and
$$2OS = e(A'Z - AZ) = e \,.\, A'A;$$

$\therefore$, letting $OA = a$,

$$OZ = \frac{a}{e} \quad \text{and} \quad OS = ae.$$

Let $P(x, y)$ be any point on the ellipse; draw PM perpendicular to DD'.

Then
$$PS^2 = e^2 \,.\, PM^2;$$

$$\therefore \; (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

i.e. $$(1-e^2)x^2+y^2=a^2(1-e^2),$$

i.e. $$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1,$$

which, if we write $b^2=a^2(1-e^2)$, becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the standard equation of the ellipse.

§ 75. *Form of the ellipse*, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

If the point (x, y) lies on the curve, the points $(x, -y)$ and $(-x, y)$ also lie on it; therefore the curve is symmetrical about both the x- and y-axes.

For real values of y, $\frac{x^2}{a^2} \ngtr 1$; therefore no part of the curve lies outside the lines $x=\pm a$.

When $x=\pm a$, $y^2=0$; therefore the lines $x=\pm a$ (Fig. 50) are tangents to the ellipse at the points $(\pm a, 0)$.

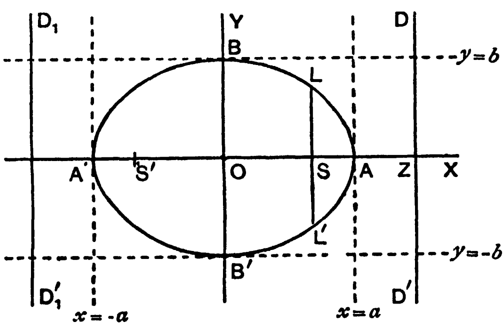

FIG. 50.

Similarly the curve does not lie outside the lines $y=\pm b$ and these lines are tangents to the curve at the points $(0, \pm b)$.

Note 1. The chord AA' passing through the focus and perpendicular to the directrix DD' is called the major axis ; O the mid-point of AA' is called the centre, and BB' the chord through the centre and perpendicular to AA' is called the minor axis. The chord LL' through the focus and perpendicular to the major axis is called the latus rectum.

In the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the centre is the origin, the major axis lies along the x-axis and has length $2a$, the minor axis lies along the y-axis and has length $2b$, while the latus rectum has length

$$2b\sqrt{1-\frac{a^2e^2}{a^2}}, \text{ i.e. } \frac{2b^2}{a}.$$

Note 2. The ellipse is symmetrical about its centre, and the centre may be defined as the point such that all chords passing through it are bisected at that point.

Note 3. From the symmetry of the curve it follows that the point $S'(-ae, 0)$ is a second focus and that the line D_1D_1' whose equation is $x=-\frac{a}{e}$ is a second directrix. The reader should establish the equation, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, taking S' and D_1D_1' as focus and directrix.

Example 1. *Find the centre, foci and directrices of the ellipse,*

$$\frac{(x+1)^2}{9}+\frac{(y-2)^2}{5}=1.$$

Changing the origin to the point $(-1, 2)$ the equation becomes

$$\frac{x^2}{9}+\frac{y^2}{5}=1 ;$$

$\therefore$ the centre is the new origin, the length of the major semi-axis is 3 and the eccentricity is e, where

$$5=9(1-e^2),$$

i.e. $$e=\tfrac{2}{3} ;$$

$\therefore$ relative to the new axes the foci are the points $(\pm 2, 0)$ and the directrices are the lines $x=\pm\frac{9}{2}$;

$\therefore$ relative to the original axes,

the centre is the point $(-1, 2)$,

the foci are the points $(1, 2)$, $(-3, 2)$,

and the directrices are the lines $x=\frac{7}{2}$, $x=-\frac{11}{2}$.

Example 2. *Find the equation of the ellipse with the point* (1, 2) *as centre, the line* $4x=29$ *as directrix and eccentricity* $\frac{4}{5}$.

Distance from centre to directrix $=6\frac{1}{4}$;

$\therefore$ length of major semi-axis $=\frac{4}{5}\times 6\frac{1}{4}=5$;

$\therefore$ square on minor semi-axis $=25(1-\frac{16}{25})=9$;

$\therefore$ required equation is

$$\frac{(x-1)^2}{25}+\frac{(y-2)^2}{9}=1,$$

which may be written

$$9x^2+25y^2-18x-100y-116=0.$$

Exercises

Find the eccentricities of the ellipses:

1. $\frac{x^2}{9}+\frac{y^2}{5}=1$. **2.** $\frac{x^2}{16}+\frac{y^2}{7}=1$. **3.** $3x^2+4y^2=12$.

Find the lengths of the latera recta of the ellipses:

4. $\frac{x^2}{9}+\frac{y^2}{4}=1$. **5.** $\frac{x^2}{5}+\frac{y^2}{3}=1$. **6.** $x^2+2y^2=4$.

Find the foci of the ellipses:

7. $\frac{x^2}{9}+\frac{y^2}{5}=1$. **8.** $\frac{x^2}{16}+\frac{y^2}{7}=1$.

9. $x^2+4y^2=4$. **10.** $2x^2-7y^2=14$.

Find the centres of the ellipses:

11. $\frac{(x-1)^2}{2}+(y-2)^2=1$. **12.** $\frac{(x+2)^2}{3}+\frac{(y-1)^2}{2}=1$.

13. $x^2+2y^2-8y+6=0$. **14.** $2x^2+3y^2-4x+12y+13=0$.

15. Find the foci of the ellipse, $\dfrac{(x+1)^2}{9}+\dfrac{(y-1)^2}{5}=1.$

16. Find the foci and directrices of the ellipse

$$\frac{(x+2)^2}{9}+\frac{(y-1)^2}{5}=1.$$

17. Find the centre and the extremities of the major axis of the ellipse

$$\frac{x^2}{25}+\frac{(y+3)^2}{9}=1.$$

Find the centres, axes and eccentricities of the ellipses:

18. $\dfrac{(x-1)^2}{25}+\dfrac{(y-2)^2}{16}=1.$ **19.** $(x+4)^2+\frac{4}{3}(y+1)^2=7.$

20. Find the centre, axes and the extremities of the major axis of the ellipse

$$\frac{(x+1)^2}{9}+\frac{(y+2)^2}{4}=1.$$

Find the centres, eccentricities, foci and directrices of the ellipses:

21. $\dfrac{(x+1)^2}{25}+\dfrac{(y-3)^2}{16}=1.$ **22.** $\dfrac{(x+1)^2}{9}+\dfrac{(2y-3)^2}{20}=1.$

23. $5x^2+9y^2-30x-36y+36=0.$

Find the equations of the ellipses from the following data:

24. Major axis $=6$, minor axis $=4$, major axis $y=2$, minor axis $x=3$.

25. Foci $(-1, 1)$, $(3, 1)$, eccentricity $\frac{2}{3}$.

26. Extremities of major axis $(-1, 1)$, $(3, 1)$, eccentricity $\frac{1}{2}\sqrt{2}$.

Sketch the ellipses:

27. $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1.$ **28.** $\dfrac{(x-3)^2}{9}+\dfrac{(y-2)^2}{4}=1.$

29. $\dfrac{(2x+1)^2}{49}+\dfrac{(y-1)^2}{4}=1.$ **30.** $4x^2+9y^2-16x-20=0.$

§ 76. *Standard equation of the hyperbola.*

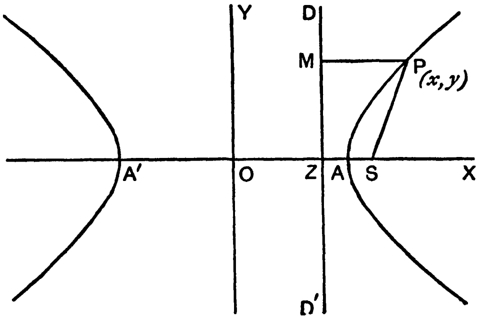

FIG. 51.

Let S (Fig. 51) be the focus and DD' the directrix. Draw SZ perpendicular to DD'. Let A, A' divide SZ internally and externally in the ratio $e:1$ where e is the eccentricity; then A, A' are points on the hyperbola. Take O the mid-point of AA' as origin, OZ as x-axis and OY perpendicular to OZ as y-axis; then

$$A'S = eA'Z,$$

and

$$AS = eZA;$$

$$\therefore\ A'A = e(A'Z - ZA) = 2e\,.\,OZ,$$

and

$$2OS = e(A'Z + ZA) = e\,.\,A'A;$$

$\therefore$, letting $OA = a$,

$$OZ = \frac{a}{e} \quad \text{and} \quad OS = ae.$$

Let $P(x, y)$ be any point on the hyperbola; draw PM perpendicular to DD'.

Then

$$PS^2 = e^2 PM^2;$$

$$\therefore\ (x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2,$$

i.e.

$$(e^2 - 1)x^2 - y^2 = a^2(e^2 - 1),$$

i.e.

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1,$$

which, if we write $b^2 = a^2(e^2 - 1)$, becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the standard equation of the hyperbola.

§ 78. *Form of the hyperbola*, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If the point (x, y) lies on the curve, the points $(x, -y)$ and $(-x, y)$ also lie on it; therefore the curve is symmetrical about both the x- and y-axes.

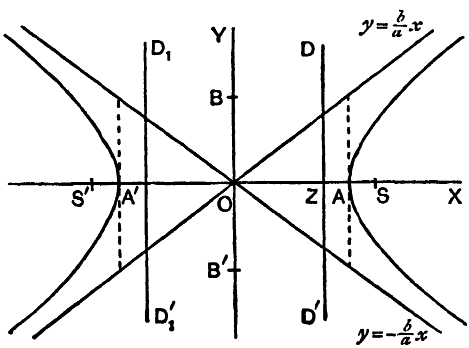

FIG. 52.

For real values of y, $\frac{x^2}{a^2} \nless 1$; therefore no part of the curve lies between the lines $x = \pm a$, but the curve extends to infinity in both directions outside these lines.

When $x = \pm a$, $y^2 = 0$; therefore the lines $x = \pm a$ (Fig. 52) are tangents to the hyperbola at the points $(\pm a, 0)$.

The equation of the hyperbola can be written,

$$y = \pm \frac{b}{a} x \sqrt{1 - \frac{a^2}{x^2}}; \qquad \text{.....................(i)}$$

therefore, as x tends to infinity, the equation tends to the equation,

$$y = \pm\frac{b}{a}x;$$

therefore the lines $y = \pm\frac{b}{a}x$ are asymptotes to the curve.

Considering equation (i) we see that, when x is large and either positive or negative, y is numerically less than $\pm\frac{b}{a}x$; hence the hyperbola approaches its asymptotes as indicated in Fig. 52.

Note 1. The equations of the asymptotes have been obtained by taking an asymptote to be a line to which the curve tends as one coordinate tends to infinity. An asymptote may be defined as a tangent to the curve at infinity, and the equations of the asymptotes may be obtained from this definition by expressing the condition that the equations $y = mx + c$ and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ should, when solved simultaneously, have two infinite roots.

Note 2. The chord AA' passing through the focus and perpendicular to the directrix DD' is called the transverse axis; O the mid-point of $A'A$ is called the centre, and BB' the line of length $2b$, which is bisected at O and which is perpendicular to $A'A$, is called the conjugate axis. It is to be observed that the conjugate axis is not a chord of the hyperbola. The chord $L'L$ through the focus and perpendicular to the transverse axis is called the latus rectum.

In the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the centre is the origin, the transverse axis lies along the x-axis and has length $2a$, the conjugate axis lies along the y-axis and has length $2b$, while the latus rectum has length

$$2b\sqrt{\frac{a^2e^2}{a^2} - 1}, \quad \text{i.e.} \quad \frac{2b^2}{a}.$$

Note 3. From the symmetry of the curve, it follows that the point $S'(-ae, 0)$ is a second focus and that the line D_1D_1', whose equation is $x = -\frac{a}{e}$, is a second directrix. The reader should establish the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, taking S' and D_1D_1' as focus and directrix.

Note 4. The hyperbola is symmetrical about its centre, and the centre may be defined as for the ellipse (see § 75, Note 2).

Example. Find the centre, eccentricity, foci, directrices and asymptotes of the hyperbola,

$$9x^2 - 16y^2 - 18x + 32y - 151 = 0.$$

The equation can be written

$$9(x^2 - 2x + 1) - 16(y^2 - 2y + 1) = 144,$$

i.e.
$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 1;$$

$\therefore$ the centre is the point (1, 1),

and the eccentricity is e, where $9 = 16(e^2 - 1)$, i.e. $e = \frac{5}{4}$;

$\therefore$ the distance from the centre to either focus is $4(\frac{5}{4})$, i.e. 5, and therefore the foci are the points (6, 1), $(-4, 1)$.

Also the distance from the centre to either directrix is $4 \div \frac{5}{4}$, i.e. $\frac{16}{5}$;

$\therefore$ the directrices are the lines, $5x = 21$, $5x + 11 = 0$.

The asymptotes are the lines

$$\frac{(x-1)^2}{16} - \frac{(y-1)^2}{9} = 0,$$

i.e.
$$3(x-1) \pm 4(y-1) = 0,$$

i.e.
$$3x + 4y - 7 = 0, \quad 3x - 4y + 1 = 0.$$

Exercises

Find the eccentricities of the hyperbolas:

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1.$ **2.** $\frac{x^2}{9} - \frac{y^2}{16} = 1.$ **3.** $8x^2 - y^2 = 2.$

Find the lengths of the latera recta of the hyperbolas:

4. $\frac{x^2}{4} - \frac{y^2}{3} = 1.$ **5.** $\frac{x^2}{9} - \frac{y^2}{2} = 1.$ **6.** $x^2 - 2y^2 = 1.$

Find the foci of the hyperbolas:

7. $\frac{x^2}{7} - \frac{y^2}{2} = 1.$ **8.** $\frac{x^2}{9} - \frac{y^2}{7} = 1.$

9. $x^2 - 8y^2 = 2.$ **10.** $9x^2 - 16y^2 = 36.$

Find the centres of the hyperbolas :

11. $\dfrac{(x+1)^2}{4}-\dfrac{(y-2)^2}{3}=1.$ **12.** $(x-1)^2-3(y+4)^2=1.$

13. $x^2-2y^2+4x+2=0.$ **14.** $x^2-2y^2-4x-4y=0.$

Find the asymptotes of the hyperbolas :

15. $\dfrac{x^2}{9}-\dfrac{y^2}{4}=1.$ **16.** $4x^2-y^2=1.$ **17.** $x^2-9y^2=3.$

Find the centres and eccentricities of the hyperbolas :

18. $\dfrac{(x+1)^2}{4}-\dfrac{(y-2)^2}{5}=1.$ **19.** $\dfrac{(x-1)^2}{9}-\dfrac{y^2}{7}=1.$

20. $(x+1)^2-2y^2=2.$ **21.** $5x^2-4y^2-10x=0.$

Find the directrices of the hyperbolas :

22. $\dfrac{x^2}{5}-\dfrac{y^2}{4}=1.$ **23.** $\dfrac{x^2}{9}-\dfrac{(y-1)^2}{7}=1.$

24. $\dfrac{(x-1)^2}{3}-(y+2)^2=1.$ **25.** $x^2-y^2+2x=1.$

Find the centres, eccentricities, foci and directrices of the hyperbolas :

26. $\dfrac{(x+1)^2}{9}-\dfrac{(y-2)^2}{7}=1.$ **27.** $\dfrac{(2x-1)^2}{4}-\dfrac{(3y-2)^2}{7}=1.$

28. $3x^2-y^2+12x+2y-1=0.$

29. Find the centre, eccentricity, foci, directrices and asymptotes of the hyperbola

$$\frac{(x+2)^2}{16}-\frac{(y+1)^2}{9}=1.$$

30. The transverse axis of a hyperbola has length 6 and is parallel to the x-axis ; the conjugate axis has length 4 ; the centre is the point (2, 3) ; find the equation of the curve.

31. Find the equation of the hyperbola with foci (5, 2), (-3, 2) and eccentricity 2.

32. Find the equation of the hyperbola with eccentricity 2, focus (1, 2) and corresponding directrix, $x+2=0$.

33. Find the equation of the hyperbola with centre (-1, 3), focus (2, 3) and eccentricity $\frac{3}{2}$.

34. Find the equation of the hyperbola with foci $(-3, -1)$, $(2, -1)$, the transverse axis being of unit length.

35. An hyperbola has foci $(6, 2)$, $(-4, 2)$ and eccentricity $\frac{5}{4}$; find the equations of the curve and its asymptotes.

Sketch the hyperbolas:

36. $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1.$

37. $\dfrac{(x+1)^2}{16} - \dfrac{(y+1)^2}{9} = 1.$

38. $x^2 - y^2 = 1.$

39. $x^2 - 4y^2 - 2x - 3 = 0.$

40. $16x^2 - 9y^2 + 32x + 36y - 164 = 0.$

CHAPTER XIV

THE PARABOLA

§ **78.** *The coordinates of any point on the parabola, $y^2 = 4ax$, expressed in terms of a single parameter.*

If we substitute $2at$ for y in the equation

$$y^2 = 4ax,$$

we have $x = at^2$; therefore the point $(at^2, 2at)$ lies on the parabola for any value of t, and may be referred to briefly as the point t.

Example 1. *The point t is one extremity of a focal chord of the parabola, $y^2 = 4ax$; prove that the other extremity is the point $-\frac{1}{t}$, and hence show that the locus of the mid-points of focal chords is the parabola*

$$y^2 = 2a(x - a).$$

Any line through the focus has equation of the form

$$y = m(x - a);$$

if this line meets the parabola at the point t,

$$2at = m(at^2 - a),$$

i.e.
$$mt^2 - 2t - m = 0.$$

The product of the roots of this equation is -1; therefore, if the point t is one extremity of the chord, the other extremity is the point $-\frac{1}{t}$.

At the mid-point of the chord, therefore,

$$2x = at^2 + \frac{a}{t^2},$$

i.e.
$$\frac{2x}{a} = t^2 + \frac{1}{t^2},$$

and
$$2y = 2at - \frac{2a}{t},$$

i.e.
$$\frac{y}{a} = t - \frac{1}{t};$$

and ∴
$$\frac{y^2}{a^2} = t^2 + \frac{1}{t^2} - 2,$$
$$= \frac{2x}{a} - 2,$$

i.e. the equation of the locus of the mid-points of focal chords is the parabola

$$y^2 = 2a(x - a).$$

Example 2. *A circle cuts the parabola, $y^2 = 4ax$, at the points t_1, t_2, t_3, t_4; show that $t_1 + t_2 + t_3 + t_4 = 0$.*

Let the circle have equation

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

The circle meets the parabola at the point t, where

$$a^2t^4 + 4a^2t^2 + 2gat^2 + 4fat + c = 0;$$

therefore this equation has roots t_1, t_2, t_3, t_4, and therefore, as the coefficient of t^3 in the equation is zero,

$$t_1 + t_2 + t_3 + t_4 = 0.$$

Exercises

1. Find the gradient of the chord joining the points t_1, t_2 on the parabola, $y^2 = 4ax$.

2. The chords PQ, PR of the parabola, $y^2 = 4ax$, are equally inclined to the axis; PQ passes through the focus; show that P and the mid-point of QR are equidistant from the axis.

3. The point t is one extremity of a focal chord of the parabola, $y^2=4ax$; show that the length of the chord is

$$a\left(t+\frac{1}{t}\right)^2.$$

4. M is the mid-point of a focal chord AB of a parabola; prove that the distance of M from the directrix is half the length of AB.

5. P, Q, R are points on the parabola, $y^2=4ax$; PQ passes through the focus, and PR is perpendicular to the axis; show that, as P moves along the given parabola, the mid-point of QR moves along the parabola

$$y^2=2a(x+a).$$

6. PQ is a chord of the parabola, $y^2=4ax$, such that the ordinate of P is twice that of Q; show that the mid-point of PQ lies on the parabola

$$5y^2=18ax.$$

7. Show that the mid-points of lines drawn from the focus to the parabola, $y^2=4ax$, lie on the parabola

$$y^2=a(2x-a).$$

8. Lines are drawn from the point $(-2a, 0)$ to the parabola, $y^2=4ax$; show that their mid points lie on the parabola

$$y^2=2a(x+a).$$

9. P is a variable point on the parabola, $y^2=4ax$, with focus S and vertex A; Q is the mid-point of SP and R is the mid-point of AQ; show that the locus of R is a parabola and find its vertex and the length of its latus rectum.

10. P is a variable point on the parabola, $y^2=4ax$, with focus S and vertex A; show that the locus of the centroid of $\triangle ASP$ is a parabola and find the coordinates of its vertex and focus.

11. PQ, a chord of the parabola $y^2=4ax$, subtends a right-angle at the vertex; show that the mid-point of PQ lies on the parabola

$$y^2=2a(x-4a).$$

12. PQ is a focal chord of the parabola, $y^2=4ax$; PR, QR are parallel to the x- and y-axes respectively; show that the locus of R has equation

$$xy^2=4a^3.$$

13. A circle cuts the parabola, $y^2=4ax$, at A, B, C, D; show that the chords AB, CD are equally inclined to the axis.

14. The coordinates of a variable point on a curve are given by the equations

$$x = at - bt^2, \quad y = ct;$$

show that the curve is a parabola, and find the vertex and the length of the latus rectum.

15. The line

$$lx + my + n = 0,$$

meets the parabola, $y^2 = 4ax$, at the points t_1, t_2; show that

$$t_1 + t_2 = -\frac{2m}{l}, \quad t_1 t_2 = \frac{n}{al},$$

and that the points coincide if

$$am^2 = ln.$$

§ 79. *Chord and Tangent.*

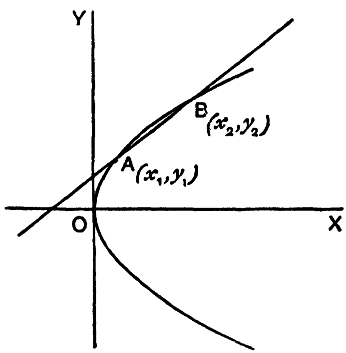

FIG. 53.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ (Fig. 53) be two points on the parabola, $y^2 = 4ax$; then

$$y_2^2 - y_1^2 = 4a(x_2 - x_1);$$

$\therefore$ the gradient of AB, which $= \dfrac{y_2 - y_1}{x_2 - x_1}$,

$$= \frac{4a}{y_1 + y_2};$$

$\therefore$ the equation of AB is

$$\frac{y-y_1}{x-x_1}=\frac{4a}{y_1+y_2},$$

i.e.
$$y(y_1+y_2)=4ax+y_1y_2. \quad \text{.................(i)}$$

Now let the point B tend to the point A; the equation (i) tends to the equation

$$2yy_1=4ax+y_1{}^2,$$

i.e. to
$$2yy_1=4ax+4ax_1,$$

i.e. to
$$yy_1=2a(x+x_1), \quad \text{..............(ii)}$$

which is therefore the equation of the tangent at A.

Using equation (i), we find that the chord joining the points t_1 and t_2 has equation

$$y=\frac{2}{t_1+t_2}x+\frac{2at_1t_2}{t_1+t_2},$$

and, using equation (ii), that the tangent at the point t has equation

$$y=\frac{x}{t}+at.$$

Note 1. Equation (i) can be established in the following way: Consider the equation

$$(y-y_1)(y-y_2)=y^2-4ax; \quad \text{....................(iii)}$$

this equation is linear and therefore represents a straight line; it is satisfied by the coordinates of A and of B and therefore represents the chord AB.

Note 2. The reader familiar with the calculus may derive the equation of the tangent thus:

$$y^2=4ax;$$

$$\therefore\ 2y\frac{dy}{dx}=4a;$$

$$\therefore\quad \frac{dy}{dx}=\frac{2a}{y};$$

$\therefore$ the tangent at (x_1, y_1) has equation

$$\frac{y-y_1}{x-x_1}=\frac{2a}{y_1},$$

i.e. $$yy_1=2ax+y_1^2-2ax_1,$$

i.e. $$yy_1=2a(x+x_1).$$

Note 3. The reader should observe that, as in the case of the circle, the equation of the tangent at a point (x_1, y_1) on a conic is obtained by substituting, in the equation of the curve, xx_1, yy_1, $\frac{1}{2}(x+x_1)$ and $\frac{1}{2}(y+y_1)$ for x^2, y^2, x and y respectively.

Note 4. From the equation of the tangent at the point t on the parabola, we see that t is the cotangent of the angle which the tangent at the point makes with the x-axis.

Example 1. *Find the mid-point of the chord,*

$$2x+y-4=0,$$

of the parabola, $y^2=4x$.

If the extremities of the given chord are the points (x_1, y_1), (x_2, y_2), the equations

$$y(y_1+y_2)=4x+y_1y_2,$$

and $$-2y=4x-8$$

represent the same line ; therefore

$$y_1+y_2=-2.$$

The ordinate of the required mid-point is therefore -1, and, from the given equation of the chord, we have that the abscissa is $\frac{5}{2}$.

Example 2. *The tangent to a parabola at a point P meets the axis at T ; prove that the tangent at the vertex bisects PT.*

Let P be the point (x_1, y_1) and let the parabola have equation

$$y^2=4ax.$$

Then the equation of the tangent at P is

$$yy_1=2a(x+x_1);$$

$\therefore$ at T $$x=-x_1;$$

$\therefore$ at the mid-point of PT

$$x=0,$$

and therefore the tangent at the vertex bisects PT.

Example 3. *Show that $P(x_1, y_1)$ is external to the parabola, $y^2=4ax$, if $y_1^2-4ax_1>0$, and deduce that there are two real tangents from any external point to the parabola.*

Take Q any point on the parabola, and let the equation of PQ be

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

Where PQ meets the parabola

$$(y_1+r\sin\theta)^2=4a(x_1+r\cos\theta),$$

i.e. $$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+(y_1^2-4ax_1)=0.$$

This equation has one real root, viz. the length of PQ; therefore the other root is real, and, if $y_1^2-4ax_1>0$, the roots have the same sign and neither root is zero, i.e. P is external to the parabola.

The tangent at the point t has equation

$$y=\frac{x}{t}+at,$$

and passes through the point (x_1, y_1), if

$$y_1=\frac{x_1}{t}+at,$$

i.e. if $$at^2-y_1t+x_1=0.$$

This equation gives two real and distinct values of t, if

$$y_1^2-4ax_1>0,$$

i.e. if the point (x_1, y_1) is external to the parabola.

There are therefore two real tangents from any external point.

§ **80.** *The line*

$$y = mx + \frac{a}{m}$$

touches the parabola, $y^2 = 4ax$, for all values of m.

Comparing the equations,

$$y = mx + c$$

and
$$y = \frac{x}{t} + at,$$

we see that the former represents the tangent at the point $\frac{1}{m}$ if

$$c = a\left(\frac{1}{m}\right).$$

Note. This value which c must have in order that the line, $y = mx + c$, may touch the parabola can be determined directly by solving simultaneously the equations of the line and curve, and expressing the condition for equal roots.

Exercises

Find the equations of the tangents to the following parabolas at the points indicated:

1. $y^2 = 4ax$, $(a, 2a)$. **2.** $y^2 = x$, $(4, -2)$.

3. $y^2 = 4(x-1)$, $(5, 4)$.

Show that the following lines are tangents to the given parabolas, and determine the points of contact:

4. $x - 6y + 9 = 0$, $y^2 = x$. **5.** $x + 2y + 4 = 0$, $y^2 = 4x$.

6. $x - 2y - 1 = 0$, $y^2 = 2(x-3)$.

7. MP is the ordinate of P, a point on the parabola, $y^2 = 4ax$; the tangent at P meets the axis at T; prove that the vertex of the parabola is the mid-point of TM.

8. The tangent at P, a point on the parabola, $y^2 = 4ax$, meets the axis at T; show that $SP = ST$ where S is the focus.

9. P is any point on a parabola with vertex A and focus S; the tangents at P and A meet at Z; prove that $SZ^2 = SA \cdot SP$.

10. Show that the line joining the origin to any point P on the parabola, $y^2 = 4ax$, has gradient twice that of the tangent at P.

11. P is a variable point on a parabola whose focus is S; the perpendicular from S to the tangent at P meets this tangent at Z; find the locus of Z.

12. PT is the tangent at $P(x_1, y_1)$, a point on the parabola, $y^2 = 4ax$; PN is parallel to the axis, and S is the focus; prove that PS and PN are equally inclined to PT and that $\tan S\hat{P}T = \pm\dfrac{2a}{y_1}$.

13. The tangent at a point on a parabola, with focus S, meets the latus rectum at A and the directrix at B; prove that $AS = BS$.

14. The tangent at a variable point on the parabola $y^2 = 4ax$, meets the axis and the tangent at the vertex at Q and R respectively; show that the locus of the mid-point of QR is the parabola

$$2y^2 + ax = 0.$$

15. PQ is a variable focal chord of a parabola; TP is the tangent at P, and TQ is parallel to the axis; show that the locus of the mid-point of PT is the directrix.

16. PT is the tangent at P, a variable point on the parabola, $y^2 = 4ax$; T lies on the directrix; show that the locus of the mid-point of PT has equation

$$y^2(2x + a) = a(3x + a)^2.$$

17. Prove that the line joining the points $(a\lambda^2, 2a\lambda)$, $(a\mu^2, 2a\mu)$ of the parabola $y^2 - 4ax = 0$ passes through the focus of the parabola, if $\lambda\mu + 1 = 0$.

Show that the radical axis of the circles described on any two focal chords of a parabola as diameters passes through the vertex of the parabola. (Camb. H.S.C.)

18. Find the equation of the chord joining the points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ on the parabola $y^2 = 4ax$.

If the chord PQ subtends a right angle at the point $(aT^2, 2aT)$ on the curve, prove that

$$(T + p)(T + q) + 4 = 0;$$

and deduce that if the circle on PQ as diameter cuts the parabola again at H and K, then the chords PQ, HK are equally inclined to the axis of the parabola. (Jt. Matric. Bd. H.S.C.)

19. Show that the tangents at the points t_1 and t_2 on the parabola, $y^2 = 4ax$, intersect at the point

$$(at_1t_2,\quad a\overline{t_1 + t_2}).$$

20. PQ, PR, two tangents to a parabola, are mutually perpendicular; show that P lies on the directrix.

21. The tangents at the extremities of a focal chord of a parabola meet at P; show that the join of P to the focus is perpendicular to the chord.

22. Prove that the tangents at the extremities of a focal chord of a parabola are mutually perpendicular.

23. P, Q are two variable points t, $2t$ on the parabola, $y^2=4ax$; TP, TQ are tangents; show that the locus of T has equation

$$2y^2=9ax.$$

24. P, Q are two variable points t_1, t_2 on the parabola, $y^2=4ax$; TP, TQ are tangents; show that, if $t_1+2t_2=0$, the locus of T has equation

$$2y^2+ax=0.$$

25. P, Q are the points t_1, t_2 on the parabola, $y^2=4ax$; TP, TQ are tangents; show that the area of $\triangle TPQ$ is $\frac{a^2}{2}(t_2-t_1)^3$.

26. P, Q are points on the parabola, $y^2=4ax$; TP, TQ are tangents; the area of $\triangle TPQ$ is $4a^2$; show that T lies on the parabola

$$y^2=4a(x+a).$$

27. A, B are the points t_1, t_2 on the parabola, $y^2=4ax$; TA, TB are tangents; the tangent at the point t meets TA, TB at C, D; show that

$$AC : TC=t-t_1 : t-t_2$$

and deduce that

$$AC : TC=TD : BD.$$

28. The tangents to a parabola from any point T are cut by the tangent at any point P, in the points Q and R. The parallel through T to the axis of the parabola cuts QR in P'. Prove that the points P and P' are equidistant from the middle point of QR.

(Oxf. H.S.C.)

29. The product of the ordinates of P, Q, two variable points on the parabola, $y^2=4ax$, is constant; show that the locus of the point of intersection of tangents at P, Q is a line parallel to the directrix.

30. P, Q are any two points on a parabola; PT, QT are tangents; M is the mid-point of PQ; prove that the parabola bisects TM.

31. PQ, a chord of the parabola, $y^2=4ax$, meets the axis at R; the tangents at P and Q meet at T; prove that R and T have equal and opposite abscissae.

32. PA, PB are tangents to a parabola with focus S; show that the inclination of PA to the axis is equal to that of PB to PS.

33. P is any point on a parabola; the chord PQ passes through the focus S, and the chord PR is perpendicular to the axis; TQ, TR are tangents; show that ST is perpendicular to the axis.

34. The chord PQ of the parabola, $y^2=4ax$, passes through the point $(b, 0)$; show that the tangents at P and Q meet on the line, $x+b=0$.

35. P is a variable point on the line, $y=mx$; tangents PQ, PR are drawn to the parabola, $y^2=4ax$; show that the locus of the mid-point of the chord QR is a parabola with vertex $\left(-\frac{a}{2m^2}, \frac{a}{m}\right)$.

36. P is a variable point on the parabola

$$y^2+4ax=0;$$

the tangents from P to the parabola, $y^2=4ax$, meet the latter at Q, R; show that the locus of the mid-point of the chord QR is the parabola

$$3y^2=4ax.$$

37. PQ is a focal chord of the parabola, $y^2=4ax$; PR, QR are parallel to the tangents at Q and P respectively; show that the locus of R is the parabola

$$y^2=a(x-3a).$$

38. The chord PQ of the parabola, $y^2=4ax$, passes through the fixed point $(-a, b)$; TP, TQ are tangents; show that T lies on the line

$$by=2a(x-a).$$

39. Find the equation of the tangent to the parabola, $y^2=4ax$, which is parallel to the line

$$3x-2y=1.$$

40. Find the equations of the tangents to the parabola, $y^2=4ax$, which are equally inclined to the axis and directrix.

41. Find the equation of the tangent to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

Obtain the equation of the tangent to this parabola which is parallel to the line $2x+y=0$, and find the coordinates of the point of contact. (Camb. H.S.C.)

42. Show that the line

$$x\cos\alpha+y\sin\alpha=p$$

touches the parabola, $y^2=4ax$, if

$$p=-a\sin\alpha\tan\alpha.$$

43. Show that the chord

$$y=mx+c$$

of the parabola, $y^2=4ax$, has length

$$\frac{4}{m^2}\sqrt{a(a-mc)(1+m^2)},$$

and deduce that the line

$$y=mx+\frac{a}{m}$$

touches the parabola.

44. Find the values of m if the line

$$y=mx+\frac{1}{m}$$

passes through the point $(-2, -1)$; hence determine the equations of the tangents from the point $(-2, -1)$ to the parabola

$$y^2=4x.$$

§ **81**. *Normal to the parabola, $y^2=4ax$.*

The tangent at (x_1, y_1) has equation, $yy_1=2a(x+x_1)$;

$\therefore$ the normal at (x_1, y_1) has equation, $y-y_1=-\dfrac{y_1}{2a}(x-x_1)$.

The tangent at the point t has equation, $y=\dfrac{x}{t}+at$;

$\therefore$ the normal at the point t has equation

$$y-2at=-t(x-at^2),$$

i.e. $$y+tx=2at+at^3.$$

Example 1. *The normal to the parabola, $y^2=4ax$, at the point t, meets the curve again at P; show that P is the point*

$$-\frac{2}{t}-t.$$

The normal has equation

$$y+tx=2at+at^3,$$

and passes through the point t_1 if

$$2at_1 + att_1^2 = 2at + at^3,$$

i.e. if $$2(t-t_1) + t(t^2 - t_1^2) = 0,$$

i.e. if $$(t-t_1)(2 + t^2 + tt_1) = 0,$$

i.e. if $$t_1 = t \quad \text{or} \quad -\frac{2}{t} - t;$$

$\therefore$ P is the point $-\frac{2}{t} - t$.

Example 2. *Show that three normals can be drawn from the point* (21, -30) *to the parabola*, $y^2 = 4x$; *find their equations.*

The normal at the point t has equation

$$y + tx = 2t + t^3,$$

and passes through the point (21, -30), if

$$-30 + 21t = 2t + t^3,$$

i.e. if $$t^3 - 19t + 30 = 0,$$

i.e. if $$(t-2)(t^2 + 2t - 15) = 0,$$

i.e. if $$t = 2,\ 3,\ -5.$$

$\therefore$ the normals at three points pass through (21, -30), and the equations of the normals are

$$y + 2x = 12, \qquad y + 3x = 33, \qquad y - 5x = -135,$$

i.e. $$2x + y - 12 = 0, \qquad 3x + y - 33 = 0, \qquad 5x - y - 135 = 0.$$

Exercises

Find the normals to the following parabolas, at the points indicated:

1. $y^2 = x$, (1, 1). **2.** $y^2 = 4x$, (1, -2).

3. $y^2 = x - 1$, (5, 2).

Show that the following lines are normals to the given parabolas, and find the coordinates of the feet of these normals:

4. $2x - y - 3 = 0$, $y^2 = x$. **5.** $2x + y - 12 = 0$, $y^2 = 4x$.

6. $2x + y - 12 = 0$, $y^2 = 2x - 6$.

7. Show that the line joining the points (1, 2), (9, −6) on the parabola

$$y^2 = 4x$$

is normal to the parabola at the former point.

8. The tangent and normal at a point on a parabola meet the axis at T and G respectively; prove that $TS = SG$ where S is the focus.

9. P is a variable point on a parabola, M the foot of the perpendicular from P to the axis and G the intersection of the axis and the normal at P; prove that MG is constant.

10. PN is the normal at P, a variable point on the parabola, $y^2 = 4ax$; the line which passes through the focus and which is parallel to the tangent at P meets PN at N; show that the locus of N is the parabola

$$y^2 = a(x - a).$$

11. M is the mid-point of PQ, a focal chord of the parabola, $y^2 = 4ax$; the line through M and parallel to the axis meets the normal at P at the point V; show that the locus of V is the parabola

$$y^2 = a(x - 3a).$$

12. P, Q, R are points on the parabola, $y^2 = 4ax$, such that PQ is perpendicular to the axis, and PR is the normal at P; the tangents at Q and R meet at T; show that the foot of the ordinate of T lies on PR.

13. P, Q are the points t, t_1 on the parabola, $y^2 = 4ax$; the normals at P, Q meet on the parabola; show that

$$tt_1 = 2.$$

14. P, Q, R are points in the parabola, $y^2 = 4ax$; PR, QR are the normals at P, Q; show that the centroid of $\triangle PQR$ lies on the axis.

15. P, Q are points on the parabola, $y^2 = 4ax$, such that the normals at P, Q intersect on the parabola; show that the mid-point of PQ lies on the parabola

$$y^2 = 2a(x + 2a).$$

16. The normal at P, a point on the parabola, $y^2 = 4ax$, meets the curve again at Q; the axis divides the chord PQ in the ratio $1 : 3$; show that P is an extremity of the latus rectum.

17. The normal at P, a point on the parabola, $y^2 = 4ax$, meets the curve again at Q; the axis divides the chord PQ in the ratio $k : 1$; show that the abscissa of P is $\dfrac{2ka}{1 - k}$.

18. Find the equation of the normal to the parabola $y^2=4ax$ at the point $P(ap^2, 2ap)$.

If this normal cuts the parabola again at the point Q, prove that the lines joining the origin to P and Q are at right angles, if $p^2=2$.

(Camb. H.S.C.)

19. Normals are drawn to the parabola, $y^2=4ax$, at the points t_1, t_2; show that the point of intersection of the normals is

$$\{a(t_1^2+t_1t_2+t_2^2+2), \quad -at_1t_2(t_1+t_2)\}.$$

20. P, Q are the points t, $\frac{2}{t}$ on the parabola, $y^2=4ax$; prove that the normals at P, Q intersect on the parabola.

21. PQ is a variable focal chord of the parabola, $y^2=4ax$; the normals at P, Q meet at N; show that the locus of N is the parabola

$$y^2=a(x-3a).$$

22. Find, in its simplest form, the equation of the normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

PQ is a chord of a parabola drawn in a fixed direction. Prove that the locus of the point of intersection of the normals at P and Q is a straight line, which is itself a normal to the parabola.

(Oxf. and Camb. H.S.C.)

23. PQ, a variable chord of the parabola, $y^2=4ax$, subtends a right angle at the vertex; the tangents at P, Q intersect at T, the normals at N, while M is the mid-point of PQ; find the loci of T, N and M.

24. PQ, a variable chord of the parabola, $y^2=4ax$, subtends a right angle at the vertex; TP, TQ are tangents; NP, NQ are normals; show that the locus of the mid-point of TN is the parabola

$$2y^2=25a(x-a).$$

25. M is the mid-point of the chord PQ of the parabola, $y^2=4ax$; PN, QN are normals at P, Q, and MN is bisected by the axis; show that the tangents at P, Q intersect on the line, $x=a$.

26. Find the equations of the three lines which pass through the point $(9a, 6a)$ and which are normal to the parabola, $y^2=4ax$.

27. Show that two of the three normals from the point $(5a, 2a)$ to the parabola, $y^2=4ax$, coincide.

28. Find the equations of the two distinct normals from the point $\left(\frac{11a}{4}, \frac{a}{4}\right)$ to the parabola, $y^2=4ax$.

29. Prove that three real normals cannot be drawn from a point $(h, 0)$ on the axis of the parabola $y^2 = 4ax$ to the curve unless $h > 2a$.

Find the area of the triangle whose vertices are the feet of the three normals from the point $(3a, 0)$. (Lond. H.S.C.)

30. Show that there are three real and distinct normals from the point (x, y) to the parabola, $y^2 = 4ax$, if

$$27ay^2 < 4(x - 2a)^3.$$

31. The normals from the point (x, y) to the parabola, $y^2 = 4ax$, meet the curve at the points (x_1, y_1), (x_2, y_2), (x_3, y_3); show that

$$y_1 + y_2 + y_3 = 0, \quad x_1 + x_2 + x_3 = 2(x - 2a).$$

32. PA, PB, PC are normals at A, B, C on the parabola, $y^2 = 4ax$; show that the circle ABC passes through the origin.

§ **82.** *Chord of contact of tangents from* $P(x_1, y_1)$ to the parabola, $y^2 = 4ax$.

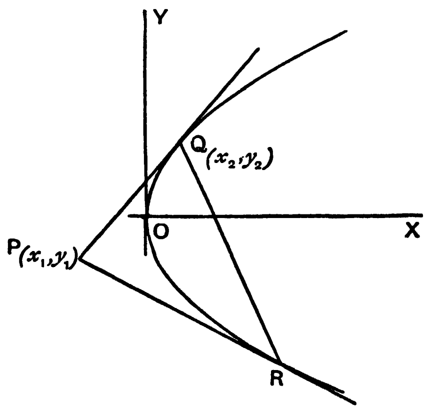

FIG. 54.

Let the tangents from P (Fig. 54) touch the parabola at Q and R.

If Q is the point (x_2, y_2), PQ has equation

$$yy_2 = 2a(x + x_2);$$

P lies on this line, therefore

$$y_1y_2 = 2a(x_1 + x_2)$$

i.e. Q lies on the line

$$yy_1 = 2a(x + x_1).$$

Similarly R lies on this line, which is therefore the line QR, i.e. the equation of the chord of contact is

$$yy_1 = 2a(x + x_1).$$

§ **83.** *Tangents from the point* $P(x_1, y_1)$ *to the parabola*, $y^2 = 4ax$.

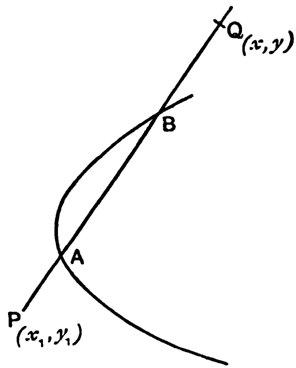

FIG. 55.

Join P (Fig. 55) to any point Q (x, y). The point dividing PQ in the ratio $k : 1$ lies on the parabola if

$$\left(\frac{ky + y_1}{k+1}\right)^2 = 4a\left(\frac{kx + x_1}{k+1}\right),$$

i.e. if $(y^2 - 4ax)k^2 + 2(yy_1 - 2a\overline{x + x_1})k + (y_1{}^2 - 4ax_1) = 0.$ ......(i)

If PQ cuts the parabola at A and B, the two values of k given by equation (i) are the values of the ratios $PA : AQ$ and $PB : BQ$. Now if Q lies on either of the tangents from P to the parabola, these two values of k must be equal and therefore

$$(y_1{}^2 - 4ax_1)(y^2 - 4ax) = (yy_1 - 2a\overline{x + x_1})^2,$$

which is therefore the equation of the tangents.

Example. *Find the equation of the pair of tangents from the point* $(-1, 2)$ *to the parabola*

$$y^2 = 2x.$$

The required equation is

$$6(y^2 - 2x) = (2y - \overline{x - 1})^2$$
$$= (2y - x + 1)^2,$$

i.e.
$$x^2 - 4xy - 2y^2 + 10x + 4y + 1 = 0.$$

§ **84.** *Definition of pole and polar.*

The polar of a point P with respect to a conic is the locus of the point of intersection of tangents drawn at the extremities of a variable chord passing through P.

The point P is called the pole of the locus.

§ **85.** *The polar of* P (x_1, y_1) *with respect to the parabola,* $y^2 = 4ax$, *is the straight line*

$$yy_1 = 2a(x + x_1).$$

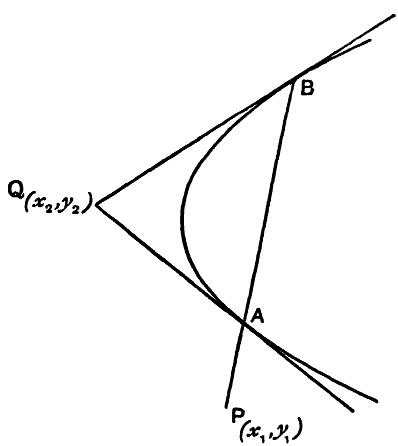

FIG. 56.

Let AB (Fig. 56), a chord of the parabola, pass through P, and

let the tangents at A and B intersect at $Q(x_2, y_2)$; then AB has equation

$$yy_2 = 2a(x + x_2);$$

P lies on this line, therefore

$$y_1y_2 = 2a(x_1 + x_2),$$

i.e. Q lies on the line

$$yy_1 = 2a(x + x_1),$$

which is therefore the polar of P.

§ **86.** *If the polar of P passes through Q, the polar of Q passes through P.*

Let P, Q be the points (x_1, y_1), (x_2, y_2).

Then the polar of P has equation

$$yy_1 = 2a(x + x_1),$$

and Q lies on this line, therefore

$$y_2y_1 = 2a(x_2 + x_1),$$

i.e. P lies on the line

$$yy_2 = 2a(x + x_2),$$

which is the polar of Q.

Note. Points such that the polar of either with respect to a conic passes through the other are called conjugate points, and lines such that either passes through the pole of the other are called conjugate lines with respect to the conic.

Example. Show that the lines

$$y = 2x + 4, \quad \text{(i)}$$

$$3y = x + 1, \quad \text{(ii)}$$

are conjugate with respect to the parabola

$$y^2 = 4x.$$

Let the pole of (i) be $P(x_1, y_1)$; then the polar of P has equation

$$yy_1 = 2(x + x_1),$$

and therefore, from (i),

$$\frac{y_1}{1}=\frac{2}{2}=\frac{2x_1}{4};$$

$\therefore$ P is the point (2, 1).

This point lies on (ii), therefore (i) and (ii) are conjugate lines.

§ **87.** *If AB, a chord of a parabola, passes through a point P and intersects the polar of P at Q, (AB, PQ) is harmonic.*

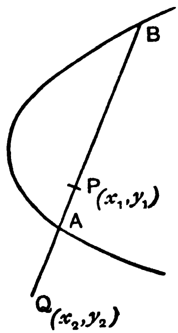

FIG. 57.

Let P, Q (Fig. 57) be the points (x_1, y_1), (x_2, y_2) and let the parabola have equation

$$y^2=4ax.$$

The point dividing PQ in the ratio $k:1$ lies on the parabola if

$$\left(\frac{ky_2+y_1}{k+1}\right)^2=4a\left(\frac{kx_2+x_1}{k+1}\right),$$

i.e. if $\quad (y_2{}^2-4ax_2)k^2+2(y_1y_2-2a\overline{x_1+x_2})k+(y_1{}^2-4ax_1)=0,$

and, since Q lies on the polar of P, the coefficient of k in this equation is zero, and therefore the two values of k which satisfy the equation are equal in magnitude and opposite in sign, i.e. A, B divide PQ internally and externally in the same ratio;

$\therefore$ (PQ, AB) is harmonic;

$\therefore$ (AB, PQ) is harmonic.

Note. If AB is a variable chord through P, the locus of Q the harmonic conjugate of P with respect to A and B is the polar of P.

Exercises

Find the equations of the chords of contact of tangents from the given points to the following parabolas ·

1. $(-2, 1)$, $y^2=x$. **2.** $(3, 2)$, $y^2+4x=0$.

3. $(0, 1)$, $y^2=x-1$.

Find the coordinates of the points, the chords of contact of tangents from which to the following parabolas are the lines given :

4. $y^2=2x$, $2y=x$. **5.** $y^2=4x$, $2x-3y+2=0$.

6. $y^2=2(x-1)$, $x-2y-3=0$.

7. Show that the chord of contact of tangents from the point $(-a, a)$ to the parabola, $y^2=4ax$, has length $5a$.

8. P is the point $(2, 3)$; PQ, PR are tangents to the parabola

$$y^2=4x\,;$$

state the equation of the chord QR, and deduce the equations of the tangents from P.

9. Show that the tangent at the point $(1, -2)$ on the parabola

$$y^2=4x$$

passes through the point $(-2, 1)$, and determine the equation of the other tangent from this point.

10. Find the equations of the tangents from the point $(-3, 1)$ to the parabola

$$y^2=x.$$

11. P, Q are points on the parabola, $y^2=4ax$; TP, TQ are tangents and cut the tangent at the vertex at A, B ; show that the line joining the focus to the mid-point of AB is perpendicular to PQ.

Find the polars of the following points with respect to the given parabolas :

12. $(-3, 4)$, $y^2=4x$. **13.** $(2, 1)$, $y^2=x$.

14. $(1, -2)$, $y^2=2x+3$.

Find the poles of the following lines with respect to the given parabolas :

15. $2x-3y+2=0$, $y^2=4x$. **16.** $3x-2y-6=0$, $y^2=3x$.

17. $3x+4y=0$, $2y^2=3(x-1)$.

Show that the following pairs of points are conjugate with respect to the given parabolas:

18. $(2, 1)$, $(2, 2)$, $y^2=x$. **19.** $(3, -1)$, $(-3, 0)$, $y^2=4x$.

20. $(-1, -2)$, $(5, -1)$, $y^2=2(x-1)$.

Show that the following pairs of lines are conjugate with respect to the given parabolas:

21. $x-y+1=0$, $2x-3y+4=0$, $y^2=4x$.

22. $3x+2y+3=0$, $x=1$, $y^2=3x$.

23. $x-2y-5=0$, $x+3y-7=0$, $y^2=2(x-3)$.

24. PQ is a focal chord of a parabola; show that the pole of PQ lies on the directrix.

25. P, Q are points on the parabola, $y^2=4ax$, such that the normals at P and Q intersect on the parabola; show that the pole of PQ lies on the line

$$x=2a.$$

26. A chord of the parabola, $y^2=4ax$, has gradient m; show that its pole lies on the line

$$y=\frac{2a}{m}.$$

27. A point lies on the line

$$2x-3y+8=0;$$

show that its polar with respect to the parabola, $y^2=4x$, passes through the point $(4, 3)$.

28. Show that the poles of the lines

$$2x-y+4a=0, \quad x-y+3a=0,$$

with respect to the parabola, $y^2=4ax$, lie on the line

$$x-y-a=0.$$

29. P, Q are points on the parabola, $y^2=4ax$, such that the normals at P, Q meet on the curve; show that the locus of the pole of PQ with respect to the circle, $x^2+y^2=a^2$, has equation

$$2x+a=0.$$

30. Tangents are drawn to the parabola, $y^2=4ax$; find the equation of the locus of their poles with respect to the circle,

$$x^2+y^2=a^2.$$

31. Normals are drawn to the parabola, $y^2=4ax$; find the locus of their poles with respect to the circle, $x^2+y^2=a^2$.

Use the reciprocal property established in § **86** to prove the following theorems for the parabola :

32. The pole of any line PQ is the point of intersection of the polars of P and Q.

33. If a number of points are collinear, their polars are concurrent.

34. If tangents are drawn from a variable point on a fixed straight line, the chords of contact are concurrent.

35. If A, B, C are points such that BC, CA are the polars of A, B, the line AB is the polar of C.

§ **88.** *The chord of the parabola, $y^2=4ax$, whose mid-point is (x_1, y_1) has equation*

$$yy_1-2ax=y_1^2-2ax_1.$$

Where the line

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \quad \text{.........................(i)}$$

meets the given parabola

$$(y_1+r\sin\theta)^2=4a(x_1+r\cos\theta),$$

i.e. $$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+y_1^2-4ax_1=0,$$

and the roots of this equation are equal in magnitude and opposite in sign if

$$y_1\sin\theta-2a\cos\theta=0;$$

$\therefore$ from (i) the required chord has equation

$$y_1(y-y_1)-2a(x-x_1)=0,$$

i.e. $$yy_1-2ax=y_1^2-2ax_1.$$

§ **89.** *Diameters of the parabola, $y^2=4ax$.*

The chord of the parabola, whose mid-point is (x_1, y_1) has equation

$$y-y_1=\frac{2a}{y_1}(x-x_1);$$

$\therefore$, if the gradient of the chord is m,

$$m=\frac{2a}{y_1},$$

i.e.
$$y_1=\frac{2a}{m};$$

$\therefore$ the mid-points of all chords of gradient m lie on the line

$$y=\frac{2a}{m}.$$

A line bisecting each of a system of parallel chords is called a diameter.

Note. Diameters of a parabola are parallel to the axis.

Example 1. *Show that the tangent at the extremity of a diameter of a parabola is parallel to the chords which the diameter bisects.*

Let the equation of the parabola be

$$y^2=4ax,$$

and let the diameter have equation

$$y=\frac{2a}{m}.$$

The extremity of this diameter is the point $\frac{1}{m}$, and the tangent at this point has equation

$$y=mx+\frac{a}{m},$$

and therefore has gradient m, i.e. the tangent is parallel to the chords which the diameter bisects.

Example 2. *Show that the equation of a parabola referred to any diameter and the tangent at its extremity as axes is*

$$y^2=4ax,$$

where a is the distance of the focus from the extremity of the diameter.

Let the parabola (Fig. 58) referred to rectangular axes OX, OY have equation, $y^2 = 4ax$, and focus S.

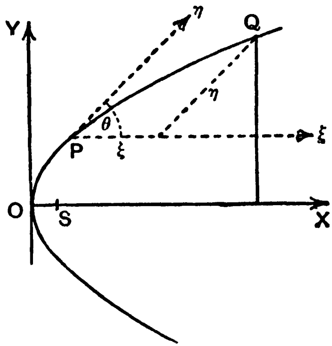

FIG. 58.

Let the tangent with gradient $\tan\theta$ touch the curve at P; then P is the point $(a\cot^2\theta,\ 2a\cot\theta)$.

Take any point $Q(x, y)$ on the curve; let its coordinates referred to the diameter and tangent through P be ξ, η.

Then $$x = a\cot^2\theta + \xi + \eta\cos\theta,$$

and $$y = 2a\cot\theta + \eta\sin\theta;$$

but $$y^2 = 4ax;$$

$$\therefore\ 4a^2\cot^2\theta + 4a\eta\cos\theta + \eta^2\sin^2\theta$$
$$= 4a^2\cot^2\theta + 4a\xi + 4a\eta\cos\theta,$$

i.e. $$\eta^2 = 4\frac{a}{\sin^2\theta}\xi.$$

Now $$SP^2 = a^2(\cot^2\theta - 1)^2 + 4a^2\cot^2\theta$$
$$= a^2(1 + \cot^2\theta)^2$$
$$= \frac{a^2}{\sin^4\theta};$$

$\therefore\ \eta^2 = 4\alpha\xi$, where the length of $SP = \alpha$.

EXERCISES

Find the equations of the chords of the following parabolas, which have the given points as mid-points :

1. $y^2=4x$, $(2, -1)$. **2.** $2y^2=3x$, $(2, -1)$.

3. $y^2=3(x+1)$, $(1, 1)$.

Find the mid-points of the following chords of the given parabolas:

4. $2x-y-2=0$, $y^2=4x$. **5.** $x+2y-1=0$, $y^2=x$.

6. $x-y-1=0$, $y^2=2(x-1)$.

7. Show that the chord

$$x-y-2a=0$$

of the parabola, $y^2=4ax$, bisects the chord

$$2x-y-5a=0.$$

Find the mid-points of the following chords of the parabola, $y^2=4ax$:

8. $lx+my=1$. **9.** $x \cos \alpha+y \sin \alpha=p$.

Find the equations of the diameters of the following parabolas, which bisect chords parallel to the given lines :

10. $y^2=4ax$, $2x-3y-a=0$. **11.** $y^2=2x$, $3x+4y+2=0$.

12. $y^2=3(x-1)$, $x+2y-1=0$.

Find the gradients of the chords of the following parabolas, which are bisected by the given diameters :

13. $y^2=4ax$, $y+4a=0$. **14.** $y^2=x$, $y+2=0$.

15. $2y^2=x-3$. $4y-1=0$.

Write down the equations of the tangents to the following parabolas at the extremities of the given diameters :

16. $y^2=4ax$, $y-2a=0$. **17.** $y^2=x$, $2y+1=0$.

18. $y^2=2(x-3)$. $y+3=0$.

19. Show that the tangents at the extremities of any chord of a parabola intersect on the diameter which bisects the chord.

20. M is the mid-point of a chord of the parabola, $y^2=4ax$; show that the chord is parallel to the polar of M.

21. M is the mid-point of a chord PQ of a parabola ; TP, TQ are tangents ; show that the parabola bisects TM.

22. Find the coordinates of the mid-point of the chord

$$y = mx + c,$$

of the parabola, $y^2 = 4ax$, and deduce that the locus of the mid-points of chords of gradient m is the diameter

$$y = \frac{2a}{m}.$$

23. PQ, a variable chord of the parabola, $y^2 = 4ax$, passes through the fixed point (x_1, y_1); show that the locus of the mid-point of PQ has equation

$$y^2 - 2ax = yy_1 - 2ax_1.$$

24. BC, a chord of the parabola, $y^2 = 4ax$, subtends a right angle at the extremity of the diameter which bisects it; show that the locus of the pole of BC has equation

$$3y^2 - 4a(x + 4a) = 0.$$

25. M is the mid-point of AB, a variable chord of the parabola, $y^2 = 4ax$; C is the extremity of the diameter through M; AB passes through a fixed point (x_1, y_1); show that the locus of the mid-point of CM has equation

$$3y^2 - 8ax = 2(yy_1 - 2ax_1).$$

CHAPTER XV

THE ELLIPSE

§ 90. In this chapter, a number of theorems concerning the ellipse are stated without proof. The reader should establish these theorems, using the methods by which the corresponding theorems concerning the parabola were established. To assist the reader in this, references to the corresponding proofs for the parabola are given in all such cases.

§ 91. *Parametric representation.*

The point $(a \cos \theta, b \sin \theta)$ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

for any value of θ.

If we write t for $\tan \dfrac{\theta}{2}$, we have that the point

$$\left(a\,\frac{1-t^2}{1+t^2},\quad b\,\frac{2t}{1+t^2}\right)$$

lies on the ellipse for any value of t.

If NP (Fig. 59), the ordinate of a point P $(a \cos \theta, b \sin \theta)$ on the ellipse, is produced to Q, so that

$$\frac{NQ}{NP}=\frac{a}{b},$$

the coordinates of Q are $a \cos \theta$, $a \sin \theta$, i.e. Q is a point on the circle

$$x^2+y^2=a^2.$$

This circle is called the auxiliary circle, and θ the vectorial angle of Q is called the eccentric angle of the point P.

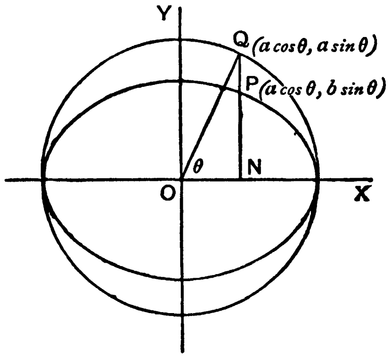

FIG. 59.

Example 1. *Show that the distances of the point θ from the foci of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are $a(1 \pm e \cos \theta)$.

Let P be the point θ, S the focus $(ae, 0)$, S' the focus $(-ae, 0)$, where e is the eccentricity.

Then
$$SP^2 = (a \cos \theta - ae)^2 + b^2 \sin^2\theta$$
$$= a^2 \cos^2\theta - 2a^2e \cos \theta + a^2e^2 + b^2 \sin^2\theta$$
$$= a^2 \cos^2\theta - 2a^2e \cos \theta + a^2 - b^2 + b^2 \sin^2\theta$$
$$= a^2 \cos^2\theta - 2a^2e \cos \theta + a^2 - b^2 \cos^2\theta$$
$$= a^2e^2 \cos^2\theta - 2a^2e \cos \theta + a^2$$
$$= a^2(1 - e \cos \theta)^2;$$
$$\therefore\ SP = a(1 - e \cos \theta).$$

Similarly $S'P = a(1 + e \cos \theta)$.

Example 2. P is a variable point θ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

Q is the point $\theta+\alpha$; show that the area of the triangle with vertices P, Q and the origin is independent of θ.

$$\begin{aligned}\text{Area of triangle} &= \tfrac{1}{2}\{a\cos\theta\,.\,b\sin(\theta+\alpha)-a\cos(\theta+\alpha)\,.\,b\sin\theta\}\\ &= \tfrac{1}{2}ab\sin\alpha,\end{aligned}$$

and is therefore independent of θ.

Exercises

1. Show that the gradient of the chord joining the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $-\dfrac{b}{a}\cot\dfrac{\theta+\phi}{2}$.

2. M is the mid-point of PQ, a variable chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with centre C; show that the product of the gradients of CM and PQ is constant.

3. AA', BB' are the major and minor axes of an ellipse with centre C; P is any point on the ellipse; PM perpendicular to AA' meets AA' at M; show that $MP^2 : A'M\,.\,MA = CB^2 : CA^2$.

4. If in the previous exercise m is the foot of the perpendicular from P to the minor axis, show that $mP^2 : B'm\,.\,mB = CA^2 : CB^2$.

5. Show that the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with eccentricity e, are the extremities of a focal chord if

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{e-1}{e+1}\ \text{ or }\ \frac{e+1}{e-1}.$$

6. The chord joining the variable points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

subtends a right angle at the point $(a, 0)$; show that

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}.$$

7. P is a variable point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with focus S; show that the locus of the mid-point of SP is an ellipse, with centre midway between the origin and S.

8. Show that the point

$$\left(a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right)$$

is the mid-point of the chord joining the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

9. Two variable points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are such that $\theta+\phi$ is constant; show that the locus of the mid-point of the chord joining the points is a straight line passing through the origin.

10. Two variable points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are such that $\theta-\phi$ is constant; show that the locus of the mid-point of the chord joining the points is an ellipse.

11. P, Q are the points θ, 2θ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

show that the locus of the mid-point of the chord PQ has equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\left(\frac{4x^2}{a^2}+\frac{4y^2}{b^2}-3\right)=\frac{x}{a}.$$

12. Show that the point whose coordinates are given by the equations

$$x=a\frac{1-t^2}{1+t^2},\quad y=\frac{2bt}{1+t^2},$$

lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 ;$$

show that the gradient of the chord joining the points t_1, t_2 is

$$-\frac{b}{a}\cdot\frac{1-t_1t_2}{t_1+t_2}.$$

13. If a circle cuts the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the points θ_1, θ_2, θ_3, θ_4, show that

$$\theta_1+\theta_2+\theta_3+\theta_4=2n\pi,$$

where n is integral. [Use the method of example 2, § 78, noting that the point θ on the ellipse is the point $\left(a\frac{1-t^2}{1+t^2}, b\frac{2t}{1+t^2}\right)$ where $t=\tan\frac{\theta}{2}$.]

14. A circle cuts the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at the points A, B, C, D; show that the chords AB, CD are equally inclined to the axes.

15. A circle meeting the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

in two coincident points at P, cuts the curve again at Q, R; show that PQ, PR are equally inclined to the axes.

16. A circle passing through the points $(a, 0)$, $(0, b)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

cuts the curve again at P, Q; show that the mid-point of PQ lies on the line

$$\frac{x}{a}+\frac{y}{b}=0.$$

§ **92.** *Chord and Tangent.*

The chord joining the points (x_1, y_1), (x_2, y_2) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has equation

$$\frac{x}{a^2}(x_1+x_2)+\frac{y}{b^2}(y_1+y_2)=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+1\,;$$

the tangent at the point (x_1, y_1) has equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

The chord joining the points θ, ϕ has equation

$$\frac{x}{a}\cos\frac{\theta+\phi}{2}+\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta-\phi}{2}\,;$$

the tangent at the point θ has equation

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1.$$

[Use the method of § **79**.]

Example 1. *MP is the ordinate of any point P on the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;$$

the tangent at P meets the major axis at T ; show that

$$OM\,.\,OT=a^2,$$

where O is the origin.

Let P be the point (x_1, y_1); then $OM=x_1$.

PT has equation, $\quad \dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}=1\,;$

$\therefore$ at T, $\qquad x=\dfrac{a^2}{x_1},$

and $\therefore$
$$OM\,.\,OT=x_1\,.\,\frac{a^2}{x_1}$$
$$=a^2.$$

Example 2. *P, Q are the points θ, ϕ on the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;$$

TP, TQ are tangents; find the coordinates of T and show that, if P and Q move so that $\theta-\phi=\frac{\pi}{2}$, the locus of T is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

TP has equation, $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$;

TQ has equation, $\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=1$;

$$\therefore \text{ at } T,\ x=a\,\frac{\sin\theta-\sin\phi}{\sin\theta\cos\phi-\cos\theta\sin\phi}=a\,\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},$$

and
$$y=b\,\frac{\cos\phi-\cos\theta}{\sin\theta\cos\phi-\cos\theta\sin\phi}=b\,\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

If $\theta-\phi=\frac{\pi}{2}$, then at T

$$x=\sqrt{2}a\cos\frac{\theta+\phi}{2},\quad y=\sqrt{2}b\sin\frac{\theta+\phi}{2};$$

$$\therefore\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=2,$$

the equation of the locus of T.

§ **93.** *The line*

$$y=mx\pm\sqrt{a^2m^2+b^2},$$

touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for all values of m.

[Use the method of § **80.**]

Example. Find the equations of the tangents common to the ellipse

$$x^2 + 2y^2 = 6,$$

and the circle

$$2x^2 + 2y^2 = 9.$$

The tangents to the ellipse, with gradient m, have equations

$$y = mx \pm \sqrt{6m^2 + 3}. \quad \text{......................(i)}$$

The tangents to the circle, with gradient m, have equations

$$y = mx \pm \frac{3}{\sqrt{2}}\sqrt{1 + m^2}. \quad \text{....................(ii)}$$

The equations (i) and (ii) represent the same lines, if

$$6m^2 + 3 = \tfrac{9}{2}(1 + m^2),$$

i.e. if

$$m = \pm 1;$$

$\therefore$ the common tangents have equations

$$y = x \pm 3,$$

and

$$y = -x \pm 3.$$

Exercises

Find the equations of the tangents to the following ellipses at the points indicated:

1. $\frac{x^2}{4} + \frac{y^2}{12} = 1, \quad (-1, 3).$ **2.** $x^2 + 2y^2 = 6, \quad (2, -1).$

3. $\frac{(x-1)^2}{2} + \frac{y^2}{8} = 1, \quad (2, -2).$

Show that the following lines are tangents to the given ellipses, and determine the points of contact:

4. $2x + 3y + 5 = 0, \quad 2x^2 + 3y^2 = 5.$

5. $2x - 2y + 3 = 0, \quad 2x^2 + 4y^2 = 3.$

6. $2x + y - 5 = 0, \quad 4(x-1)^2 + (y+1)^2 = 8.$

7. Show that the line

$$x \cos \alpha + y \sin \alpha = p$$

touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

if
$$p=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}.$$

8. Show that the line

$$lx+my+n=0,$$

touches the elllipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

if
$$a^2l^2+b^2m^2=n^2.$$

9. Show that the line

$$y-k=m(x-h)\pm\sqrt{a^2m^2+b^2},$$

is a tangent to the ellipse

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1.$$

10. If m is the gradient of a common tangent to the circle

$$x^2+y^2=c^2,$$

and the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

show that
$$m^2=\frac{c^2-b^2}{a^2-c^2};$$

hence find the common tangents to the circle

$$x^2+y^2=25,$$

and the ellipse
$$\frac{x^2}{169}+\frac{y^2}{16}=1.$$

11. P is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

S, S' are the foci; show that SP, $S'P$ are equally inclined to the tangent at P.

12. The tangent at P, a point on an ellipse, meets the directrix at T; show that PT subtends a right angle at the focus.

13. If p, p_1 are the distances of the foci from a variable tangent to an ellipse, show that pp_1 is equal to the square on the minor semi-axis.

14. P is a point on the circle

$$x^4+y^2=a^2;$$

Q is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

having the same abscissa as P; show that the tangents at P and Q meet on the x-axis.

15. P is a point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

the tangent at P and the perpendicular from P to the minor axis meet this axis at t and m respectively; prove that

$$Om\,.\,Ot=b^2,$$

where O is the origin.

16. P, Q are the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

TP, TQ are tangents; show that T is the point

$$\left(a\cos\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2},\quad b\sin\frac{\theta+\phi}{2}\sec\frac{\theta-\phi}{2}\right).$$

17. P, Q are points on an ellipse; TP, TQ are tangents; show that the line joining T to the mid-point of the chord PQ passes through the centre of the ellipse.

18. P, Q are variable points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

the mid-point of the chord PQ lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k;$$

show that the tangents at P, Q intersect on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{k}.$$

19. P, Q are the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

TP, *TQ* are tangents; show that the area of $\triangle TPQ$ is

$$ab\frac{\sin^3\dfrac{\theta-\phi}{2}}{\cos\dfrac{\theta-\phi}{2}}.$$

20. Show that the line $x\cos\alpha+y\sin\alpha=p$ touches the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

if
$$p^2=a^2\cos^2\alpha+b^2\sin^2\alpha.$$

If the perpendiculars from B, B', the extremities of the minor axis, to any tangent are equal to p_1, p_2 in length, prove that the locus of Q is a circle through B, B', where Q is distant p_1, p_2 from B and B' respectively. (Lond. H.S.C.)

21. Show that any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

may be denoted by

$$\frac{x}{a}=\frac{1-t^2}{1+t^2},\quad \frac{y}{b}=\frac{2t}{1+t^2},$$

and find the equation of the chord joining the points t_1 and t_2.

The lines joining any point on the ellipse to the points $(\lambda a, 0)$, $(\lambda' a, 0)$ meet the ellipse again at Q, R. Prove that the tangents at Q, R meet on the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\,\frac{(1-\lambda^2)(1-\lambda'^2)}{(1-\lambda\lambda')^2}=1.$$

Interpret the result when $\lambda\lambda'=1$. (Oxf. and Camb. H.S.C.)

22. Prove that any point whose coordinates x and y satisfy the equations

$$\frac{1-\dfrac{x}{a}}{t^2}=\frac{1+\dfrac{x}{a}}{1}=\frac{\dfrac{y}{b}}{t},$$

where t is an auxiliary variable, lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and that the coordinates of the point of intersection of the tangents to the ellipse at the points for which $t=t_1$ and $t=t_2$ satisfy the equations

$$\frac{1-\frac{x}{a}}{t_1t_2}=\frac{1+\frac{x}{a}}{1}=\frac{\frac{y}{b}}{\frac{1}{2}(t_1+t_2)}.$$

ABC is any triangle circumscribing an ellipse. A', B', C' are the points of the ellipse diametrically opposite the points of contact of the sides BC, CA, AB respectively. Prove that AA', BB', CC' are concurrent. (Oxf. H.S.C.)

23. P, Q are the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

the chord PQ touches the circle

$$x^2+y^2=b^2\ ;$$

show that the tangents at P and Q intersect on the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{b^2}.$$

24. P, Q are points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

TP, TQ are tangents to the ellipse; the chord PQ touches the parabola

$$y^2=4cx\ ;$$

show that T lies on the parabola

$$a^2cy^2+b^4x=0.$$

25. TP, TQ are tangents at P, Q the extremities of a focal chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

show that T lies on the directrix.

26. A variable chord PQ of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

subtends a right angle at the point $(a, 0)$; show that PQ passes through a fixed point on the major axis.

Find the equations of the tangents to the following ellipses, which are parallel to the given lines:

27. $\dfrac{x^2}{5}+\dfrac{y^2}{4}=1, \quad x-y+1=0.$

28. $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1, \quad 2x-y+3=0.$

29. $\dfrac{(x-1)^2}{2}+\dfrac{y^2}{7}=1, \quad x+y-2=0.$

§ **94.** *Normal to the ellipse,* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1.$

The normal at the point (x_1, y_1) has equation

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2;$$

the normal at the point θ has equation

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

[Use the method of § **81**.]

Example 1. *Find the condition that the line*

$$lx+my=1,$$

should be normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equations $\quad \dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2,$

and $\quad lx+my=1$

represent the same line if

$$\frac{a}{l}=(a^2-b^2)\cos\theta \quad\text{and}\quad -\frac{b}{m}=(a^2-b^2)\sin\theta;$$

$\therefore$ if

$$\frac{a^2}{l^2}+\frac{b^2}{m^2}=(a^2-b^2)^2,$$

which is therefore the required condition.

Example 2. *Show that the feet of the normals from the point* (x_1, y_1) *to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

lie on the curve

$$(a^2-b^2)xy+b^2y_1x-a^2x_1y=0.$$

The normal at a point (ξ, η) on the ellipse has equation

$$\frac{a^2x}{\xi}-\frac{b^2y}{\eta}=a^2-b^2.$$

If this normal passes through the point (x_1, y_1),

$$\frac{a^2x_1}{\xi}-\frac{b^2y_1}{\eta}=a^2-b^2,$$

i.e. the point (ξ, η) lies on the curve

$$\frac{a^2x_1}{x}-\frac{b^2y_1}{y}=a^2-b^2,$$

i.e. $$(a^2-b^2)xy+b^2y_1x-a^2x_1y=0.$$

Example 3. *The normals at the points* θ_1, θ_2, θ_3, θ_4 *on the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are concurrent; *show that*

$$\theta_1+\theta_2+\theta_3+\theta_4=(2n+1)\pi,$$

when n is integral.

The normal at the point θ passes through the point (x_1, y_1) if

$$\frac{ax_1}{\cos\theta}-\frac{by_1}{\sin\theta}=a^2-b^2,$$

i.e. if $$ax_1\sin\theta-by_1\cos\theta=(a^2-b^2)\sin\theta\cos\theta. \quad \ldots\ldots\text{(i)}$$

Writing t for $\tan\frac{\theta}{2}$, equation (i) is

$$\{2ax_1t-by_1(1-t^2)\}(1+t^2)=(a^2-b^2)2t(1-t^2),$$

i.e. $$by_1t^4+2(ax_1+a^2-b^2)t^3+2(ax_1-a^2+b^2)t-by_1=0. \quad \ldots\ldots\text{(ii)}$$

If the normals at the points θ_1, θ_2, θ_3, θ_4 pass through the point (x_1, y_1), equation (ii) will have roots

$$\tan\frac{\theta_1}{2},\quad \tan\frac{\theta_2}{2},\quad \tan\frac{\theta_3}{2},\quad \tan\frac{\theta_4}{2};$$

$$\therefore \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2} = -1,$$

and
$$\Sigma \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} = 0;$$

but
$$\tan\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}$$

$$= \frac{\Sigma\tan\frac{\theta_1}{2} - \Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}}{1-\Sigma\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2}};$$

$\therefore$
$$=\infty;$$

$$\therefore \frac{\theta_1+\theta_2+\theta_3+\theta_4}{2} = (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is integral};$$

$$\therefore \theta_1+\theta_2+\theta_3+\theta_4 = (2n+1)\pi, \text{ where } n \text{ is integral.}$$

Exercises

Find the normals to the following ellipses at the points indicated:

1. $2x^2+y^2=6$, $(1, 2)$. **2.** $3x^2+2y^2=5$, $(-1, 1)$.

3. $\dfrac{(x-1)^2}{2}+y^2=3$, $(3, -1)$.

Show that the following lines are normals to the given ellipses, and find the coordinates of the feet of these normals:

4. $2x+y-2=0$, $x^2+2y^2=12$.

5. $4x+3y-2=0$, $3x^2+2y^2=11$.

6. $x-y-1=0$, $\dfrac{(x-3)^2}{4}+\dfrac{y^2}{12}=1$.

7. Find the length of a diameter POQ of an ellipse of centre O and semi-axes a, b which makes an angle $\frac{1}{4}\pi$ with the major axis.

Prove also that the tangent of the angle between OP and the normal to the ellipse at P is $\frac{a^2-b^2}{a^2+b^2}$. (Jt. Matric. Bd. H.C.)

8. Find the condition that the line

$$y=mx+c$$

should be normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

9. P is a point on an ellipse with focus S and eccentricity e; the normal at P meets the major axis at G; show that $GS=eSP$.

10. P is the point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

the normal at P meets the y-axis at g; show that the ordinate of g is

$$-e^2\frac{a^2}{b^2}y_1.$$

11. P is the point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

the normal at P cuts the x-axis at G; prove that the abscissa of G is e^2x_1, where e is the eccentricity.

12. P is any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with foci S, S'; the normal at P meets the minor axis at N; show that

$$NS^2 : SP \,.\, S'P=a^2-b^2 : b^2.$$

13. Show that the coordinates of the point of intersection of normals at the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are

$$\frac{a^2-b^2}{a}\cos\theta\cos\phi\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\quad -\frac{a^2-b^2}{b}\sin\theta\sin\phi\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}.$$

14. NP, NQ are normals to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

show that, if PQ is a focal chord, N has the same ordinate as the mid-point of the chord.

15. Show that the mid-point of the chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

which is the normal at the point θ, lies on the line

$$\frac{x}{a^3}\cos\theta+\frac{y}{b^3}\sin\theta=0.$$

16. The normals at the points θ_1, θ_2, θ_3 on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are concurrent; show that

$$\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)+\sin(\theta_1+\theta_2)=0.$$

§ **95.** *The chord of contact of tangents from the point* (x_1, y_1) *to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

has equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

[Use the method of § **82.**]

§ **96.** *The equation of the pair of tangents from the point* (x_1, y_1) *to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is
$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2.$$

[Use the method of § **83.**]

Example. Show that the locus of the point of intersection of mutually perpendicular tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the circle

$$x^2+y^2=a^2+b^2.$$

(This circle is called the director circle.)

The tangents from the point (x_1, y_1) to the ellipse have equation

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2,$$

and, if these lines are perpendicular,

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}-1\right)=0,$$

i.e. $$x_1^2+y_1^2=a^2+b^2,$$

i.e. the point of intersection of the tangents lies on the circle

$$x^2+y^2=a^2+b^2,$$

which is therefore the required locus.

§ **97**. *Pole and Polar.*

The polar of a point with respect to a conic has already been defined in § **84**.

The polar of the point (x_1, y_1) with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is the line

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

[Use the method of § **85**.]

Following the methods of §§ **86** and **87**, the reader should prove that

(i) if the polar of P, with respect to an ellipse, passes through Q, the polar of Q passes through P,

and (ii) if AB, a chord of an ellipse, passes through a point P and intersects the polar of P at Q, (AB, PQ) is harmonic.

Example. Show that the locus of the poles of chords which are normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has equation

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

The polar of the point (x_1, y_1) has equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1. \qquad \text{(i)}$$

The normal at the point θ has equation

$$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2. \qquad \text{(ii)}$$

Equations (i) and (ii) represent the same line if

$$\frac{x_1\cos\theta}{a^3}=-\frac{y_1\sin\theta}{b^3}=\frac{1}{a^2-b^2},$$

i.e. if

$$\frac{a^6}{x_1^2}+\frac{b^6}{y_1^2}=(a^2-b^2)^2;$$

$\therefore$ if (x_1, y_1) is the pole of a normal chord, (x_1, y_1) lies on the curve

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

Exercises

Find the equations of the chords of contact of tangents from the given points to the following ellipses:

1. $(2, 1)$, $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$.

2. $(-1, 1)$, $2x^2+5y^2=3$.

3. $(1, 0)$, $\dfrac{(x+1)^2}{2}+(y-1)^2=1$.

Find the coordinates of the points, the chords of contact of tangents from which to the following ellipses are the lines given :

4. $\dfrac{x^2}{2}+y^2=1$, $x+y-1=0$. **5.** $3x^2+4y^2=2$, $3x-4y-2=0$.

6. $\dfrac{(x-2)^2}{4}+y^2=1$, $x+8y-6=0$.

7. State the equation of the chord of contact of tangents from the point $(4, 6)$ to the ellipse

$$\frac{x^2}{16}+\frac{y^2}{12}=1,$$

and deduce the equations of the tangents from the given point to the ellipse.

8. Find the equations of the tangents from the point $(1, 2)$ to the ellipse

$$x^2+2y^2=6.$$

9. Find the equation of the pair of tangents from the point $(2, -1)$ to the ellipse

$$2x^2+3y^2=4.$$

10. Show that the tangents from the point $(2, 1)$ to the ellipse

$$2x^2+3y^2=6$$

are mutually perpendicular.

Find the polars of the following points with respect to the given ellipses :

11. $(3, 1)$, $x^2+3y^2=15$. **12.** $(-2, 1)$, $3x^2+4y^2=18$.

13. $(1, -\frac{1}{2})$, $\dfrac{(2x-3)^2}{4}+(y+1)^2=1$.

Find the poles of the following lines with respect to the given ellipses :

14. $4x+y-1=0$, $4x^2+3y^2=3$. **15.** $x-2y+4=0$, $x^2+2y^2=8$.

16. $x-6y-4=0$, $x^2+4y^2+4y=0$.

17. Find the equation of the polar of the point $(2t, 1-t)$ with respect to the ellipse

$$ax^2+by^2=1,$$

and show that, for all values of t, the polar passes through a fixed point; determine the coordinates of this point.

Show that the following points are conjugate with respect to the given ellipses:

18. $(2, 1), (-1, 2), \quad x^2+2y^2=2.$

19. $(-1, 1), (1, 2), \quad 3x^2+4y^2=5.$

20. $(0, -2), (-3, 1), \quad 4x^2+3y^2-4x=0.$

Show that the following lines are conjugate with respect to the given ellipses:

21. $x+2y-1=0, \quad x-y+2=0, \quad 5x^2+6y^2=15.$

22. $x-2y-6=0, \quad x+y-1=0, \quad x^2+4y^2=12.$

23. $2x-y+2=0, \quad 3x+4y-4=0, \quad 4x^2+9y^2=12(x+y).$

24. Show that the polar of a focus of an ellipse is the corresponding directrix.

25. Show that the poles of the lines

$$x+2y+3a=0, \quad 2x+3y+4a=0,$$

with respect to the ellipse

$$2x^2+3y^2=4a^2,$$

lie on the line

$$x-3y-2a=0.$$

26. Show that each pair of the lines

$$x+6y-a=0,$$
$$5x-3y+a=0,$$
$$x-6y+11a=0,$$

are conjugate with respect to the ellipse

$$x^2+3y^2=a^2.$$

27. A chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

has gradient m; show that its pole lies on the line

$$y=-\frac{b^2}{a^2m}x.$$

28. The polar of P with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meets the x-axis at Q; the line through P perpendicular to the polar meets the x-axis at R; show that, if O is the origin,

$$OQ \, . \, OR = a^2 - b^2.$$

29. Show that the points which lie on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2,$$

and which are distant c from their polars with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

lie on the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}.$$

30. The line through P perpendicular to the polar of P with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

meets the x- and y-axes at Q, R; V is the fourth vertex of the rectangle with vertices Q, R and the origin; show that, if P lies on the line

$$\frac{x}{a^2} + \frac{y}{b^2} = 1,$$

V lies on the line

$$x - y = a^2 - b^2.$$

31. A variable chord PQ of an ellipse passes through a fixed point. P', Q' are the points of the ellipse diametrically opposite P and Q. Prove that the pole, with respect to the ellipse, of $P'Q'$ lies on a fixed straight line. (Oxf. H.S.C.)

32. Prove that the perpendicular from any point P of the conic $(a^2 - b^2)xy + b^2y_0x - a^2x_0y = 0$ to the polar line of P with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ passes through the point (x_0, y_0), and deduce that four normals can be drawn from any point to an ellipse.

If the tangents to the ellipse at the feet of two of these normals meet in the point (x_1, y_1), prove that the line joining the feet of the other two normals is $\frac{x}{x_1} + \frac{y}{y_1} + 1 = 0$. amb. H.S.C.)

§ **98.** *The chord of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

whose mid-point is (x_1, y_1) *has equation*

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}.$$

[Use the method of § **88**.]

§ **99.** *Diameters of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The chord whose mid-point is (x_1, y_1) has equation

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2};$$

$\therefore$ if the gradient of the chord is m,

$$m=-\frac{b^2}{a^2}\cdot\frac{x_1}{y_1},$$

i.e.
$$y_1=-\frac{b^2}{a^2m}x_1;$$

$\therefore$ the mid-points of all chords of gradient m lie on the line

$$y=-\frac{b^2}{a^2m}x.$$

This is a line passing through the centre of the ellipse and is called a diameter of the ellipse.

§ **100.** *Conjugate diameters of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

In § **99** it is shown that the mid-points of all chords of gradient m lie on the diameter

$$y=-\frac{b^2}{a^2m}x;$$

∴ the mid-points of all chords of gradient $-\frac{b^2}{a^2m}$ lie on the diameter

$$y=-\frac{b^2}{a^2}\left(-\frac{a^2m}{b^2}\right)x,$$

i.e. on the diameter $$y=mx.$$

Hence, if a diameter A bisects chords parallel to a diameter B, the diameter B bisects chords parallel to A. Two such diameters are said to be conjugate. The reader should remember that conjugate diameters have the product of their gradient equal to $-\frac{b^2}{a^2}$.

Example 1. *A, B are the points θ, ϕ on the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with centre O; show that OA, OB are conjugate semi-diameters if

$$\theta\sim\phi=\frac{\pi}{2}.$$

OA, OB have gradients $\frac{b}{a}\tan\theta$, $\frac{b}{a}\tan\phi$;

∴ OA, OB are conjugate semi-diameters if

$$\tan\theta\tan\phi=-1,$$

and this relation holds when $\theta\sim\phi=\frac{\pi}{2}$.

Example 2. *AB is a diameter of an ellipse; show that the tangents at A, B are parallel to the diameter conjugate to AB.*

Let A be the point θ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

then B is the point $(\theta+\pi)$, and the tangents at A, B have equations

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\quad -\frac{x\cos\theta}{a}-\frac{y\sin\theta}{b}=1.$$

Each of these tangents has gradient $-\frac{b}{a}\cot\theta$;

and the gradient of AB is $\frac{b}{a}\tan\theta$;

$\therefore$ the tangents are parallel to the diameter conjugate to AB.

Example 3. *Prove that the tangents at the extremities of a chord of an ellipse intersect on the diameter which bisects the chord.*

The tangents at the points θ, ϕ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

intersect at the point

$$\left(a\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}},\ b\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}}\right). \quad \text{..................(i)}$$

The chord joining the points θ, ϕ has mid-point

$$\left(a\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},\ b\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\right). \quad \text{.........(ii)}$$

The points (i) and (ii) lie on the line

$$\frac{y}{x}=\frac{b}{a}\tan\frac{\theta+\phi}{2};$$

$\therefore$ the tangents intersect on the diameter bisecting the chord.

Exercises

Find the chords of the following ellipses which have the given points as mid-points:

1. $x^2+3y^2=8$, $(2, 1)$.
2. $3x^2+4y^2=12$, $(-1, 1)$.
3. $x^2+2y^2-2x=0$, $(\frac{3}{2}, -\frac{1}{2})$.

Find the mid-points of the given chords of the following ellipses:

4. $3x-5y-8=0$, $3x^2+5y^2=15$.

5. $2x-4y+3=0, \quad x^2+2y^2=1.$

6. $2y=x+2, \quad (x-1)^2+2y^2=7.$

Find the equations of the diameters of the following ellipses, which bisect chords parallel to the given lines :

7. $x^2+2y^2=1, \quad 2x-3y+1=0.$

8. $3x^2+4y^2=2, \quad x+2y-3=0.$

9. $(x-1)^2+3y^2=1, \quad x+y-2=0.$

Find the gradients of the chords which are bisected by the given diameters of the following ellipses :

10. $x=2y, \quad x^2+2y^2=2.$ 11. $9x+8y=0, \quad 3x^2+4y^2=1.$

12. $x-3y-2=0, \quad \frac{(x-2)^2}{3}+2y^2=1.$

13. CP, CD are two equal and conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

show that their gradients are $\pm\frac{b}{a}$.

14. CP, CD are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

p is the distance of P from CD ; show that

$$CD=\frac{ab}{p}.$$

15. OC, OD are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

show that the mid-point of the chord CD lies on the ellipse

$$2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=1.$$

16. Show that tangents at the extremities of conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

intersect on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2.$$

17. *OC*, *OD* are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

TC, *TD* are tangents; show that the area *OCTD* is *ab*.

18. *OC*, *OD* are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

NC, *ND* are normals; show that *CD* is perpendicular to the line joining *N* to the origin.

19. *AB* is a fixed diameter of an ellipse; *CP*, *CD* are conjugate semi-diameters; *PM*, *DN* are the perpendiculars to *AB*; show that PM^2+DN^2 is constant.

20. Show that, for all values of θ, the extremities of the chord

$$\left(\frac{x}{a}+\frac{y}{b}\right)\cos\theta-\left(\frac{x}{a}-\frac{y}{b}\right)\sin\theta=1$$

of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are the extremities of conjugate semi-diameters.

21. *OC*, *OD* are conjugate semi-diameters of an ellipse with foci S, S_1; prove that $CS\,.\,CS_1=OD^2$.

22. *CP*, *CQ* are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

the tangent at *P* meets the *x*-axis at *A*, the tangent at *Q* the *y*-axis at *B*; show that the mid-point of *AB* lies on the lines

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0.$$

23. The extremities of a variable chord of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are the extremities of conjugate semi-diameters; show that the locus of the pole of the chord with respect to the circle

$$x^2+y^2=c$$

is the ellipse

$$a^2x^2+b^2y^2=2c^2.$$

24. P, Q are extremities of conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;$$

the chord PQ meets the x- and y-axes at A, B; show that the locus of the mid-point of AB has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=8\,\frac{x^2y^2}{a^2b^2}.$$

25. Define *conjugate diameters* of an ellipse, and find the equation of the diameter of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

conjugate to the diameter $\qquad y=mx.$

If ϕ, ϕ' are the eccentric angles of the extremities of a pair of conjugate semi-diameters, show that

$$\phi\sim\phi'=\frac{\pi}{2}.$$

If PSQ is a focal chord of an ellipse, whose centre is at C, and CP, CQ are conjugate semi-diameters, prove that if ϕ is the eccentric angle of P, then

$$e(\cos\phi+\sin\phi)=1,$$

and PQ equals the semi-major axis. (Cent. Welsh Bd. H.S.C.)

§ 101. *Ellipse as projection of a circle.*

Let P (Fig. 60) be any point (x, y) on the circle whose equation is

$$x^2+y^2=a^2,$$

relative to axes OX, OY, and let NP be the ordinate of P. Let $O\eta$ and Np be the projections of OY and NP on the plane

passing through OX and inclined to the plane of the circle at $\cos^{-1}\dfrac{b}{a}$.

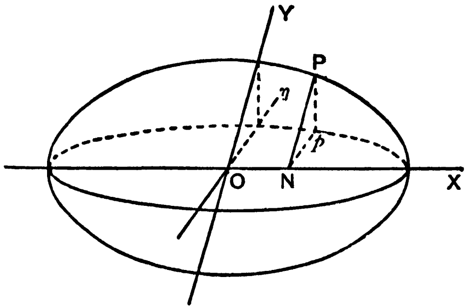

FIG. 60.

Then, if p is the point (ξ, η) relative to axes OX, $O\eta$,

$$\xi = x,$$

and $$\eta = y\,\frac{b}{a};$$

but $$x^2 + y^2 = a^2;$$

$$\therefore\ \xi^2 + \frac{a^2}{b^2}\eta^2 = a^2,$$

i.e. $$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = 1;$$

∴ the projection of the circle is an ellipse with semi-axes a and b.

Note. Many properties of the ellipse can be established by considering the ellipse as the projection of a circle. Before attempting to answer the exercises at the end of this paragraph, the reader should follow carefully the worked examples, remembering that

(i) the projections of parallel lines are themselves parallel,

and (ii) the ratio of the projections of two parallel lines is the ratio of the lines.

Example 1. *NP is the ordinate of P, a point on the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with centre O; the tangent at P meets the major-axis at T; show that $ON.OT=a^2$.

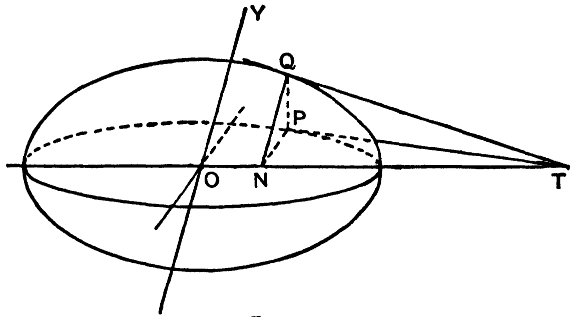

FIG. 61.

Consider the ellipse (Fig. 61) as the projection of the circle

$$x^2+y^2=a^2;$$

let Q be the point of which P is the projection.

TP meets the ellipse in two coincident points at P; therefore TQ meets the circle in two coincident points at Q, i.e. TQ is the tangent to the circle at Q;

$$\therefore\ ON.OT=OQ^2$$
$$=a^2.$$

Example 2. *Show that the locus of the mid-points of parallel chords of an ellipse is a diameter.*

The parallel chords of the ellipse are the projections of parallel chords of a circle; the mid-points of the chords of the ellipse are the projections of the mid-points of the chords of the circle. The latter mid-points all lie on a diameter of the circle, and the projection of a diameter of the circle is a diameter of the ellipse; hence the mid-points of the parallel chords of the ellipse lie on a diameter of the ellipse.

Example 3. *Show that, if a diameter A of an ellipse bisects chords parallel to a diameter B, B bisects chords parallel to A.*

Consider the ellipse as the projection of a circle, and let A, B be the projections of the diameters a, b of the circle. Then A bisects chords parallel to B; therefore a bisects chords parallel to b. The diameters a, b are therefore mutually perpendicular, and therefore b bisects chords parallel to a. Hence B bisects chords of the ellipse which are parallel to A.

Example 4. *H, K are any two points on an ellipse with a diameter PQ; PH, QK meet at X; PK, QH meet at Y; show that XY is parallel to the diameter conjugate to PQ.*

Consider the ellipse as the projection of a circle, and let h, k, p, q, x, y (Fig. 62) be the points of which H, K, P, Q, X, Y are the projections.

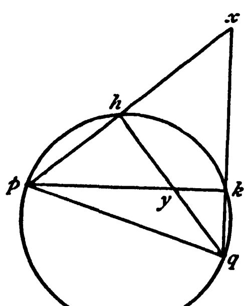

FIG. 62.

Then pq is a diameter of the circle; therefore pk, qh are perpendiculars of $\triangle xpq$. Therefore xy is perpendicular to pq, and hence XY is parallel to the diameter of the ellipse conjugate to PQ.

Exercises

1. P is any point on an ellipse with diameter QR; prove that PQ, PR are parallel to conjugate diameters.

2. PP', DD' are conjugate diameters of an ellipse; show that the diameters which bisect PD and $P'D$ are conjugate.

3. CP, CD are conjugate semi-diameters of an ellipse; the tangent at P meets the axes at Q, R; show that $QP.PR = CD^2$.

4. M is the mid-point of PQ, a chord of an ellipse with centre C; show that the tangents at P and Q intersect on CM produced; show also that, if the tangents intersect at T and CT meets the ellipse at A, $CM.CT = CA^2$.

5. Q is any point on an ellipse with conjugate diameters PP', DD'; PQ, $P'Q$ meet DD' at M, N; show that $CM.CN = CD^2$, where C is the centre of the ellipse.

6. PQ is a focal chord of an ellipse; P', Q' are the points on the auxiliary circle corresponding to P, Q; show that the tangents at P', Q' to the circle intersect on the directrix.

7. P is any point on an ellipse with a diameter QR; PR meets the tangent at Q in T; show that the tangent at P bisects QT.

8. T is any point on the tangent at P to an ellipse with centre C; the line through P parallel to TC meets the ellipse at Q, and R is the point diametrically opposite to Q; show that TR is the tangent at R.

9. PQ, HK are two chords of an ellipse, such that PQ is bisected by HK; the tangents at P, Q meet at T and the tangents at H, K meet at V; show that TV is parallel to PQ.

10. M is the mid-point of a chord HK of an ellipse with centre C; CM meets the ellipse at P; N is the mid-point of the chord PH; CN, HM intersect at Q; show that CH bisects the chord passing through P and Q.

11. CP is any semi-diameter of an ellipse with diameter QCR; the line through Q parallel to CP meets RP at V and the ellipse at K; show that the tangent at P bisects VK.

12. SR, a chord of an ellipse, is bisected by the diameter PQ; the tangent at S meets at T the line through R parallel to QS; show that PS bisects RT.

13. CP, CQ are conjugate semi-diameters of an ellipse; R is the other extremity of the diameter through Q; the tangent at any point T meets QR at V and CP at X; the line through R parallel to CT meets CP at Y; show that RX and the parallel through Q to VY intersect on the ellipse.

14. CP, CQ are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

a tangent parallel to PQ meets CP, CQ at R, S; show that R and S lie on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2;$$

show also that the line RS is bisected at the point of contact.

15. A tangent from the point $(2c, 0)$ to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meets the curve at P; show that P lies on the ellipse

$$\frac{(x-c)^2}{a^2}+\frac{y^2}{b^2}=\frac{c^2}{a^2}.$$

16. Prove that the sum of the squares on two sides of a triangle is twice the square on the median through their point of intersection plus twice the square on half the base.

PQ is a variable diameter of an ellipse, A is a fixed point in its plane. The lines AP, AQ meet the ellipse again at R, S respectively. Prove, by orthogonal projection, that $\dfrac{AP}{AR}+\dfrac{AQ}{AS}$ is constant.

(Oxf. and Camb. H.S.C.)

CHAPTER XVI

THE HYPERBOLA

§ 102. *Parametric representation.*

The point $(a \sec \theta, b \tan \theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

for any value of θ.

If we write t for $\tan \frac{\theta}{2}$, we have that the point

$$\left(a \frac{1+t^2}{1-t^2},\ b \frac{2t}{1-t^2}\right)$$

lies on the hyperbola for any value of t.

Example. *P, Q are the points θ, ϕ on the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $\theta + \phi = \frac{\pi}{2}$; show that the locus of the mid-point of the chord PQ has equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{y}{b}.$$

At the mid-point of the chord PQ,

$$2x = a(\sec \theta + \sec \phi)$$
$$= a(\sec \theta + \operatorname{cosec} \theta),$$

and

$$2y = b(\tan \theta + \tan \phi)$$
$$= b(\tan \theta + \cot \theta),$$

and $\therefore$
$$4\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=2\sec\theta\operatorname{cosec}\theta$$
$$=2\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$=2(\tan\theta+\cot\theta)$$
$$=\frac{4y}{b};$$

$\therefore$ the equation of the locus is
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{y}{b}.$$

Exercises

1. Show that the gradient of the chord joining the points θ, ϕ on the hyperbola
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
is
$$\frac{b\cos\frac{\theta-\phi}{2}}{a\sin\frac{\theta+\phi}{2}}.$$

2. Show that the gradient of the line joining the origin to the mid-point of the chord joining the points t_1, t_2 on the hyperbola
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
is
$$\frac{b}{a}\cdot\frac{t_1+t_2}{1+t_1t_2}.$$

3. Show that the line joining the points t_1, t_2 on the hyperbola
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$
has gradient
$$\frac{b}{a}\cdot\frac{1+t_1t_2}{t_1+t_2}.$$

4. Show that the coordinates of the mid-point of the chord joining the points θ, ϕ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

can be written

$$a \sec\theta \sec\phi \cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}, \quad b \sec\theta \sec\phi \sin\frac{\theta+\phi}{2}\cos\frac{\theta+\phi}{2}.$$

5. P, Q are the points $\frac{\pi}{4}+\theta$, $\frac{\pi}{4}-\theta$ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that the locus of the mid-point of PQ has equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{y}{b}.$$

6. The points θ, ϕ are the ends of a focal chord of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=\frac{1-e}{1+e} \quad \text{or} \quad \frac{1+e}{1-e}.$$

7. P, Q are variable points θ, $-\theta$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with major axis AA'; AP, $A'Q$ meet at R; show that the locus of R is the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

8. P is a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

Q is the mid-point of the line joining P to the vertex $(a, 0)$; show that the locus of Q is a hyperbola with centre $\left(\frac{a}{2}, 0\right)$ and axes half the lengths of the axes of the given hyperbola.

9. P is the point (x, y) on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

with foci S, S'; show that

$$SP=ex-a,$$

and that
$$S'P=ex+a.$$

10. Show that the difference of the focal distances of a point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is constant and equal to the length of the transverse axis.

§ **103.** *Chord and Tangent.*

The chord joining the points (x_1, y_1), (x_2, y_2) on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

has equation

$$\frac{x}{a^2}(x_1+x_2)-\frac{y}{b^2}(y_1+y_2)=\frac{x_1x_2}{a^2}-\frac{y_1y_2}{b^2}+1\ ;$$

the tangent at the point (x_1, y_1) has equation

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

The chord joining the points θ, ϕ has equation

$$\frac{x}{a}\cos\frac{\theta-\phi}{2}-\frac{y}{b}\sin\frac{\theta+\phi}{2}=\cos\frac{\theta+\phi}{2};$$

the tangent at the point θ has equation

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

[Use the method of § **79.**]

Example. The tangent at P, a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

meets the axes OX, OY at A, B respectively, and $OAQB$ is a rectangle; show that the locus of Q has equation

$$\frac{a^2}{x^2}-\frac{b^2}{y^2}=1.$$

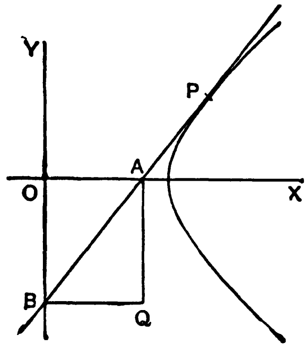

FIG. 63.

Let P (Fig. 63) be the point θ; then PAB has equation

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1;$$

therefore, at A, $\quad x=a\cos\theta$,

and, at B, $\quad y=-b\cot\theta$;

$\therefore$ Q is the point $(a\cos\theta,\ -b\cot\theta)$, and therefore the equation of its locus is

$$\frac{a^2}{x^2}=1+\frac{b^2}{y^2},$$

i.e.
$$\frac{a^2}{x^2}-\frac{b^2}{y^2}=1.$$

§ **104.** The line

$$y=mx\pm\sqrt{a^2m^2-b^2}$$

touches the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

for all values of m.

[Use the method of § **80**.]

EXERCISES

Find the equations of the tangents to the following hyperbolas at the points indicated:

1. $\frac{x^2}{8} - y^2 = 1$, $(4, 1)$.

2. $x^2 - 2y^2 = 1$, $(-3, 2)$.

3. $(x-2)^2 - y^2 = 3$, $(0, -1)$.

Show that the following lines are tangents to the given hyperbolas, and determine the points of contact:

4. $x + 1 = 0$, $4x^2 - 3y^2 = 4$.

5. $x - 2y + 1 = 0$, $x^2 - 6y^2 = 3$.

6. $x + y - 1 = 0$, $\frac{(2x-3)^2}{5} - y^2 = 1$.

7. Show that the line

$$x \cos a + y \sin a = \sqrt{a^2 \cos^2 a - b^2 \sin^2 a}$$

is a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

8. Show that the line

$$lx + my + n = 0$$

touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

if

$$a^2l^2 - b^2m^2 = n^2.$$

9. The tangent at P, any point on a hyperbola, meets the directrix at T; show that PT subtends a right angle at the focus.

10. NP is the ordinate of P, any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

the tangent at P meets the transverse axis at T; show that

$$ON \,.\, OT = a^2,$$

where O is the origin.

11. P is any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

the tangent at P meets the conjugate axis at T', and PN' perpendicular to the conjugate axis meets this axis at N'; show that

$$ON' . OT' = -b^2,$$

where O is the origin.

12. P is a point on a hyperbola; the line joining P to the focus is parallel to an asymptote; prove that the asymptote, the directrix and the tangent at P are concurrent.

13. P is a variable point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with focus S; PT is the tangent at P and ST is perpendicular to PT; show that the locus of T is the circle

$$x^2 + y^2 = a^2.$$

14. Show that the point of intersection of the tangents at the points θ, ϕ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has coordinates

$$a \frac{\cos \frac{\theta - \phi}{2}}{\cos \frac{\theta + \phi}{2}}, \quad b \tan \frac{\theta + \phi}{2}.$$

15. P, Q are the points θ, $\frac{\pi}{2} + \theta$ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

show that the tangents at P, Q intersect on the hyperbola

$$\frac{2x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

16. P, Q are the points θ, $\frac{\pi}{2} - \theta$ on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1;$$

show that the tangents at P, Q intersect on the line

$$y = b.$$

17. Show that the point $\left(a\dfrac{1+t^2}{1-t^2},\ b\dfrac{2t}{1-t^2}\right)$ lies on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,;$$

show that the equation of the chord joining the points t_1, t_2 is

$$\frac{x}{a}(1+t_1t_2)-\frac{y}{b}(t_1+t_2)=1-t_1t_2.$$

18. P is the point t_1 in the previous exercise, Q is the point t_2; show that, if $t_1+t_2=c$, the tangents at P and Q intersect on the line

$$2ay=bc(x+a).$$

19. Prove that *any* point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

may be represented by

$$x=\frac{a}{2}\left(t+\frac{1}{t}\right),\quad y=\frac{b}{2}\left(t-\frac{1}{t}\right),$$

and find the equation of the tangent at t.

The point P on a hyperbola, with focus S, is such that the tangent at P, the latus rectum through S and one asymptote are concurrent. Prove that SP is parallel to the other asymptote.

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20. Find the equations of the tangents to the hyperbola

$$2x^2-3y^2=6,$$

which are parallel to the line

$$x+y-2=0.$$

21. Find the equations of the tangents common to the hyperbola

$$\frac{x^2}{4}-\frac{y^2}{2}=1$$

and the circle

$$x^2+y^2=1.$$

§ **105.** *Normal to the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The normal at the point (x_1, y_1) has equation

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2\,;$$

the normal at the point θ has equation

$$\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2.$$

[Use the method of § **81**.]

Exercises

Find the equations of the normals to the following hyperbolas at the points indicated:

1. $x^2-2y^2=2, \quad (-2, 1).$ **2.** $3x^2-2y^2=1, \quad (1, -1).$

3. $3(x-1)^2-4y^2=18, \quad (4, \frac{3}{2}).$

Show that the following lines are normals to the given hyperbolas, and find the points at which the lines are normals:

4. $3x+4y-10=0, \quad 2x^2-3y^2=5.$

5. $5x-8y+18=0, \quad 4x^2-5y^2=11.$

6. $2x-3y+8=0, \quad (2x-1)^2-2y^2=1.$

7. P is a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\ ;$$

the normal at P meets the axes OX, OY at A, B respectively, and $OAQB$ is a rectangle; show that the locus of Q is the hyperbola

$$a^2x^2-b^2y^2=(a^2+b^2)^2.$$

8. The normal at P, a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

meets the conjugate axis at Q; QP is produced its own length to R: show that the locus of R is the hyperbola

$$\frac{x^2}{4a^2}-\frac{b^2y^2}{(a^2-b^2)^2}=1.$$

9. Show that the normal at the point θ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

meets the curve again at the point ϕ, where

$$a^2\sin\theta\sin\frac{\theta+\phi}{2}+b^2\cos\frac{\theta-\phi}{2}=0.$$

10. Show that the poles of normals to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

with respect to the circle

$$x^2+y^2=a^2+b^2,$$

lie on the curve

$$\frac{a^2}{x^2}-\frac{b^2}{y^2}=1.$$

11. Normals are drawn to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,;$$

show that the locus of their poles with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has equation

$$\frac{a^6}{x^2}-\frac{b^6}{y^2}=(a^2+b^2)^2.$$

§ **106.** *The chord of contact of tangents from the point* (x_1, y_1) *to the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

has equation

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

[Use the method of § **82**.]

§ **107.** *The equation of the pair of tangents from the point* (x_1, y_1) *to the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is

$$\left(\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1\right)\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1\right)^2.$$

[Use the method of § **83**.]

§ **108.** *Pole and Polar.*

The polar of the point (x_1, y_1) with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is the line

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1.$$

[Use the method of § **85**.]

Note. The reader should note that theorems (i) and (ii) of § **97** hold for the hyperbola.

Example. Show that the poles of tangents to the parabola

$$ay^2-2b^2x=0,$$

with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

lie on the parabola

$$ay^2+2b^2x=0.$$

The polar of (x_1, y_1) with respect to the hyperbola is the line

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1,$$

i.e.
$$y=\frac{b^2x_1}{a^2y_1}x-\frac{b^2}{y_1}$$

and, if this line is a tangent to the parabola

$$ay^2-2b^2x=0,$$

then
$$\left(\frac{b^2x_1}{a^2y_1}\right)\left(-\frac{b^2}{y_1}\right)=\frac{b^2}{2a};$$

i.e.
$$ay_1^2+2b^2x_1=0,$$

i.e. (x_1, y_1) lies on the parabola

$$ay^2+2b^2x=0.$$

Exercises

Find the equations of the chords of contact of tangents from the given points to the following hyperbolas:

1. $(1, 2)$, $2x^2-3y^2=1$. **2.** $(2, -4)$, $x^2-2y^2=4$.

3. $(0, -2)$, $(x-1)^2-3(y+1)^2=2$.

4. Prove that the tangents to a hyperbola at the extremities of a focal chord intersect on the directrix.

5. Write down the equation of the chord of contact of tangents from the point $(1, -2)$ to the hyperbola

$$\frac{x^2}{3}-\frac{y^2}{2}=1\ ;$$

hence determine the equations of the tangents.

6. Find the equation of the pair of tangents from the point $(1, -1)$ to the hyperbola

$$2x^2-3y^2=6.$$

7. The tangents from P to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

are mutually perpendicular; show that the locus of P is the circle

$$x^2+y^2=a^2-b^2.$$

8. Show that the points $(1, 2)$, $(-2, -1)$, $(-1, -\frac{2}{3})$ are the vertices of a triangle which is self-conjugate with respect to the hyperbola

$$2x^2-3y^2=2.$$

9. Show that the poles of tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

lie on the ellipse.

10. Show that the locus of the poles of tangents to the parabola

$$y^2=4cx,$$

with respect to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

is the parabola

$$a^2cy^2+b^4x=0.$$

§ **109.** *The chord of the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

whose mid-point is (x_1, y_1) *has equation*

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=\frac{x_1{}^2}{a^2}-\frac{y_1{}^2}{b^2}.$$

[Use the method of § **88**.]

§ **110.** *Diameters of the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The mid-points of all chords of gradient m lie on the diameter

$$y=\frac{b^2}{a^2m}x.$$

[Use the method of § **89**.]

§ **111.** *Conjugate diameters of the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

If a diameter A bisects chords parallel to a diameter B, the diameter B bisects chords parallel to A.

[Use the method of § **100**.]

Two such diameters are said to be conjugate. The reader should remember that conjugate diameters have the product of their gradients equal to $\frac{b^2}{a^2}$.

§ **112.** *Conjugate hyperbolas.*

Two hyperbolas are said to be conjugate when the transverse and conjugate axes of one are respectively the conjugate and transverse axes of the other.

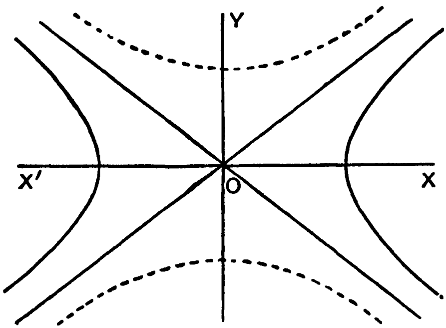

FIG. 64.

The hyperbola which is conjugate to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

has equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$$

for the equation of the conjugate hyperbola (Fig. 64) referred to OY as x-axis and OX' as y-axis is

$$\frac{x^2}{b^2}-\frac{y^2}{a^2}=1,$$

and on reverting to the axes OX, OY this equation becomes

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1,$$

i.e.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

Two conjugate hyperbolas have therefore the same asymptotes.

§ **113.** *Conjugate diameters (continued).*

If a diameter of a hyperbola meets the curve in real points, the conjugate diameter does not.

The diameter, $y=mx$, meets the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

where

$$\frac{x^2}{a^2}-\frac{m^2x^2}{b^2}=1,$$

i.e. where

$$x=\pm\frac{ab}{\sqrt{b^2-a^2m^2}},$$

and therefore x is real and finite, if and only if

$$m<\frac{b}{a} \text{ numerically.}$$

If two diameters, $y=m_1x$, $y=m_2x$, are conjugate,

$$m_1m_2=\frac{b^2}{a^2};$$

$\therefore$ if one of the gradients m_1, m_2 is numerically less the other is numerically greater than $\frac{b}{a}$, and therefore if one diameter meets the curve in real points the other does not.

The diameter, $y=mx$, meets the conjugate hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1,$$

in real points at a finite distance if

$$m>\frac{b}{a} \text{ numerically};$$

therefore, of two conjugate diameters of a hyperbola, one intersects the hyperbola, the other the conjugate hyperbola.

The points of intersection are called the extremities of the conjugate diameters.

Example 1. *Show that the point P $(a \tan \theta, b \sec \theta)$ lies on the hyperbola conjugate to the hyperbola,*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \; ; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

show that the points

$$Q(a \sec \theta, b \tan \theta), \quad P(a \tan \theta, b \sec \theta)$$

are the extremities of conjugate semi-diameters of (i).

At P,
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \tan^2\theta - \sec^2\theta$$
$$= -1 \; ;$$

$\therefore$ P lies on the conjugate hyperbola.

If O is the centre of (i),

$$\text{gradient of } OP = \frac{b \sec \theta}{a \tan \theta},$$

and
$$\text{gradient of } OQ = \frac{b \tan \theta}{a \sec \theta} \; ;$$

$\therefore$ OP, OQ are conjugate, the product of their gradients being $\frac{b^2}{a^2}$.

Example 2. *Show that a hyperbola referred to its asymptotes as axes of co-ordinates has equation of the form*

$$xy = \textit{constant}.$$

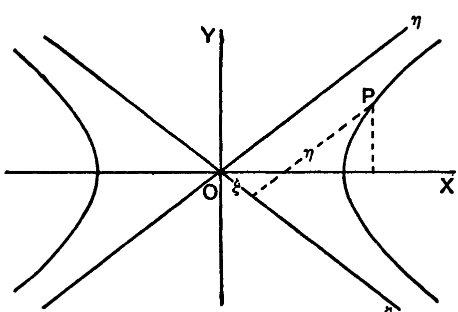

Fig. 65.

Let the hyperbola (Fig. 65) referred to rectangular axes OX, OY have equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Let P, the point (x, y), have co-ordinates ξ, η referred to the asymptotes $O\xi$, $O\eta$, and let $X\hat{O}\eta=\theta$.

Then $$x=(\xi+\eta)\cos\theta, \qquad y=(-\xi+\eta)\sin\theta;$$

$$\therefore \frac{x}{a}=\frac{\xi+\eta}{\sqrt{a^2+b^2}}, \qquad \frac{y}{b}=\frac{-\xi+\eta}{\sqrt{a^2+b^2}};$$

but $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

$$\therefore (\xi+\eta)^2-(-\xi+\eta)^2=a^2+b^2,$$

i.e. $$\xi\eta=\frac{a^2+b^2}{4}.$$

Exercises

1. CP, CQ are conjugate semi-diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that the mid-point of the line PQ lies on an asymptote.

2. CP, CQ are conjugate semi-diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that $CP^2 \sim CD^2=a^2-b^2$.

3. Show that the line joining the extremities of two conjugate semi-diameters of a hyperbola is parallel to one of the asymptotes.

4. Show that the tangents at the extremities of a diameter of a hyperbola are parallel to the conjugate diameter.

5. Show that the tangents at the extremities of two conjugate diameters of a hyperbola intersect on the asymptotes.

6. OP, OQ are conjugate semi-diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that the area of $\triangle OPQ$ is $\frac{1}{2}ab$.

7. CP, CQ are semi-diameters of a hyperbola; CP', CQ' are the conjugate semi-diameters; show that PQ, $P'Q'$ are parallel to each other or to conjugate diameters.

8. CP, CQ are semi-diameters of a hyperbola; CP', CQ' are the conjugate semi-diameters; M, M' are the mid-points of the chords PQ, $P'Q'$; show that CM, CM' lie along conjugate diameters.

9. P is a point on an asymptote of a hyperbola; show that the polar of P is parallel to the asymptote on which P lies.

10. Show that the polar of P with respect to a hyperbola touches the conjugate hyperbola if P lies on the latter.

11. P, Q are the extremities of conjugate semi-diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

PQ meets the x-axis at R; QV, RV are parallel to the x- and y-axes respectively; show that the co-ordinates of V satisfy the equation

$$\frac{x^2}{a^2}\pm\frac{2xy}{ab}=\pm 1.$$

12. P is a point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

Q is a point on the conjugate hyperbola, such that P and Q are the extremities of conjugate semi-diameters; QP is produced its own length to R; show that the locus of R is the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=3.$$

§ 114. *Rectangular hyperbola.*

A hyperbola is said to be rectangular when its transverse and conjugate axes are equal in length.

The equation

$$x^2-y^2=a^2$$

therefore represents a rectangular hyperbola referred to its axes as axes of co-ordinates.

In the case of the rectangular hyperbola, the equation referred to its asymptotes as axes, viz.:

$$xy = \text{a constant}$$

is a specially useful one as the axes are rectangular. Writing the equation

$$xy = c^2,$$

the point $\left(ct, \frac{c}{t}\right)$ lies on the curve for any value of t, and we have the following equations:

(i) the chord joining the points (x_1, y_1), (x_2, y_2),

$$c^2x + x_1x_2y = c^2(x_1 + x_2);$$

(ii) the chord joining the points t_1, t_2,

$$x + t_1t_2y = c(t_1 + t_2);$$

(iii) the tangent at the point (x_1, y_1),

$$xy_1 + x_1y = 2c^2;$$

(iv) the tangent at the point t,

$$x + t^2y = 2ct;$$

(v) the normal at the point (x_1, y_1),

$$xx_1 - yy_1 = x_1^2 - y_1^2;$$

(vi) the normal at the point t,

$$tx - \frac{y}{t} = c\left(t^2 - \frac{1}{t^2}\right),$$

(vii) the chord of contact of tangents from the point (x_1, y_1),

$$xy_1 + x_1y = 2c^2;$$

(viii) the tangents from the point (x_1, y_1),

$$4(x_1y_1 - c^2)(xy - c^2) = (xy_1 + x_1y - 2c^2)^2;$$

(ix) the polar of the point (x_1, y_1),

$$xy_1 + x_1y = 2c^2;$$

(x) the chord with mid-point (x_1, y_1),

$$xy_1 + x_1y = 2x_1y_1\,;$$

(xi) the diameter bisecting chords of gradient m,

$$y = -mx.$$

Two diameters of the hyperbola, $xy = c^2$, are conjugate if their gradients are equal in magnitude but opposite in sign.

Example 1. *A variable chord of the rectangular hyperbola*

$$xy = c^2$$

is such that its projection on the x-axis has constant length $2c$; *show that the locus of its mid-point has equation*

$$x^2y = c^2(x+y).$$

Let the chord have extremities (x_1, y_1), (x_2, y_2); then, at the mid-point

$$2x = x_1 + x_2 = \pm\sqrt{(x_1 - x_2)^2 + 4x_1x_2} = \pm\sqrt{4c^2 + 4x_1x_2},$$

and $$2y = y_1 + y_2 = c^2\left(\frac{1}{x_1} + \frac{1}{x_2}\right) = c^2\frac{x_1 + x_2}{x_1x_2} = c^2\frac{2x}{x_1x_2},$$

and therefore $$4x^2 - 4c^2 = 4\frac{c^2x}{y},$$

i.e. $$x^2y = c^2(x+y),$$

the required equation.

Example 2. *A straight line passing through the fixed point* (α, β) *meets the x- and y-axes at P, Q respectively*; *show that the locus of the mid-point of PQ is a rectangular hyperbola with centre* $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

Let PQ (Fig. 66) have equation

$$\frac{x}{2a} + \frac{y}{2b} = 1\,;$$

then $$\frac{\alpha}{2a} + \frac{\beta}{2b} = 1,$$

and at M $$x=a, \quad y=b;$$

$\therefore$ at M, $$\frac{a}{2x}+\frac{\beta}{2y}=1,$$

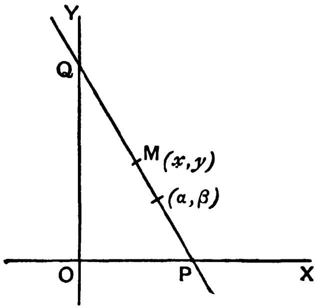

FIG. 66.

i.e. the locus of M has equation

$$xy-\frac{\beta}{2}x-\frac{a}{2}y=0,$$

i.e. $$\left(x-\frac{a}{2}\right)\left(y-\frac{\beta}{2}\right)=\frac{a\beta}{4}. \quad \text{.........................(i)}$$

Referred to parallel axes through $\left(\frac{a}{2}, \frac{\beta}{2}\right)$, equation (i) becomes

$$xy=\frac{a\beta}{4},$$

and therefore represents a rectangular hyperbola with centre at the point $\left(\frac{a}{2}, \frac{\beta}{2}\right)$ referred to the original axes.

EXERCISES

1. Show that the equation

$$xy+2x-y-6=0$$

represents a rectangular hyperbola, and find the equation of its axes.

2. Show that the equation

$$y=\frac{ax+b}{cx+d},$$

where a, b, c, d are constants, represents a rectangular hyperbola with centre $\left(-\frac{d}{c}, \frac{a}{c}\right)$.

3. Write down the general equation of a straight line passing through the point (a, b).

A line is drawn through the point $(1, 2)$ to cut the axes OX, OY at H, K respectively, and the parallelogram $OHLK$ is completed. Find the equation of the locus of L. (Jt. Matric. Bd. H.S.C.)

4. P is the point $\{a \tan\theta, b \tan(\theta+\phi)\}$, where θ is variable and ϕ is constant; show that the locus of P is a rectangular hyperbola with centre $(a \cot\phi, -b \cot\phi)$.

5. Find the gradient of the chord joining the points t_1, t_2 as the hyperbola, $xy=c^2$; hence or otherwise show that the orthocentre of the triangle whose vertices are the points t_1, t_2, t_3 is the point $-\frac{1}{t_1t_2t_3}$.

6. P, Q are two variable points on the hyperbola, $xy=c^2$, such that the tangent at Q passes through the foot of the ordinate P; show that the locus of the mid-point of the chord PQ is a hyperbola with the same asymptotes as the given hyperbola.

7. A variable chord of the hyperbola, $xy=c^2$, passes through the fixed point (a, b); show that the locus of the mid-point of the chord is a rectangular hyperbola and determine its asymptotes.

8. Show that the line

$$x+y+1=0$$

is a tangent to the hyperbola

$$4xy=1,$$

and to the parabola

$$y^2=4x;$$

determine the points of contact.

9. P is a variable point on the hyperbola, $xy=c^2$; the tangent at P cuts the x- and y-axes at A, B respectively; Q is the fourth vertex of the rectangle with vertices A, B and the origin; show that the locus of Q is another hyperbola with the same asymptotes as the given one.

10. Find in its simplest form the condition that the four points with co-ordinates $\left(\kappa t, \frac{\kappa}{t}\right)$, where the parameter t has the values a, b, c, d, should lie on a circle.

A, B, C, D are four points on a rectangular hyperbola and are not concyclic; if the circles BCD, CAD, ABD, ABC meet the rectangular hyperbola again in the points α, β, γ, δ respectively, prove that the middle points of the chords $A\alpha$, $B\beta$, $C\gamma$, $D\delta$ lie on another rectangular hyperbola with the same asymptotes. (Camb. H.S.C.)

11. P, Q are the points t_1, t_2 on the hyperbola, $xy=c^2$; show that the gradient of the chord PQ is $-\frac{1}{t_1 t_2}$; hence or otherwise prove that, if PQ subtends a right angle at a third point R on the hyperbola, the tangent at R is perpendicular to PQ.

12. Show that the tangent at the point t on the hyperbola, $xy=c^2$, has equation

$$x+t^2y=2ct.$$

NP is the ordinate of a point P on the hyperbola; the tangent at P meets the y-axis at M, and the line through M parallel to the x-axis meets the hyperbola at Q; show that NQ is the tangent at Q.

13. Show that the tangents at the points $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola, $xy=c^2$, intersect at the point

$$\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right).$$

14. P, Q are the points $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola, $xy=c^2$; TP, TQ are tangents; show that the line joining the origin to T bisects the chord PQ.

15. P, Q are variable points $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola $xy=t$; show that, if t_1t_2 is constant, the locus of the intersection of tangents at P and Q is a straight line passing through the origin.

16. P, Q are the points $\left(ct_1, \frac{c}{t_1}\right)$, $\left(ct_2, \frac{c}{t_2}\right)$ on the hyperbola, $xy=c^2$; TP, TQ are tangents, M is the mid-point of the chord PQ, and $AMBT$ is a rectangle with sides parallel to the axes of co-ordinates; show that A and B lie on the hyperbola.

17. P, Q are two variable points on the hyperbola, $xy=c^2$, such that the tangent at Q passes through the foot of the ordinate of P; show that the locus of the intersection of tangents at P and Q is a hyperbola with the same asymptotes as the given hyperbola.

18. The normal at $P(x_1, y_1)$, a point on the hyperbola, $xy=c^2$, meets the curve again at $Q(x_2, y_2)$; show that

$$x_1x_2=-y_1^2, \quad y_1y_2=-x_1^2.$$

19. The normal at P, a variable point on the hyperbola, $xy=c^2$, meets the curve again at Q; show that the locus of the mid-point of PQ has equation

$$4x^3y^3+c^2(x^2-y^2)^2=0.$$

20. P, Q are the points $\left(ct, \frac{c}{t}\right)$, $\left(ct_1, \frac{c}{t_1}\right)$ on the hyperbola, $xy=c^2$; PQ is normal to the curve at P; show that

$$t^3t_1=-1.$$

21. A variable normal to the hyperbola $xy=c^2$ meets the x- and y-axes at P, Q respectively; show that the locus of the mid-point of PQ has equation

$$c^2(x^2-y^2)^2+4x^3y^3=0.$$

22. The normal at a variable point P on the hyperbola, $xy=c^2$, meets the x-axis at Q; show that the locus of the mid-point of PQ has equation

$$8y^4=c^2(c^2-2xy).$$

CHAPTER XVII

POLAR EQUATIONS OF CONICS

§ 115. *The polar equation of a conic, the focus being the pole.*

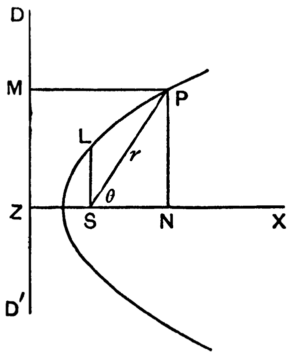

FIG. 67.

Let S (Fig. 67) be the focus and DD' the directrix. Let XSZ be perpendicular to DD', and let SL, the semi-latus rectum, have length l. Let P be any point on the conic and let PM, PN be perpendicular to DD', ZX.

Take SX as the positive direction of the initial line, and let P have co-ordinates r, θ; then, if the eccentricity of the conic is e, we have

$$\begin{aligned} r &= SP \\ &= e \,.\, MP \\ &= e(ZS + SN) \\ &= SL + er\cos\theta \\ &= l + er\cos\theta, \end{aligned}$$

i.e. $$\frac{l}{r}=1-e\cos\theta, \qquad \text{(i)}$$

which is therefore the polar equation of the conic.

Note 1. The reader should verify that, in the case of the farther branch of a hyperbola, equation (i) holds if r is the negative radius vector corresponding to the vectorial angle θ.

Note 2. If SZ is chosen as the initial line, the polar equation is

$$\frac{l}{r}=1+e\cos\theta.$$

Here also r is negative for points on the farther branch of a hyperbola.

Note 3. If the axis of the conic is inclined at angle α to the initial line, the polar equation is

$$\frac{l}{r}=1\mp e\cos(\theta-\alpha),$$

according as the initial line is SX or SZ.

Note 4. In the case of the parabola, equation (i) becomes

$$\frac{l}{r}=1-\cos\theta$$
$$=2\sin^2\frac{\theta}{2}.$$

Example. P, Q are the extremities of a focal chord of a conic with focus S and semi-latus rectum of length l; show that

$$\frac{1}{SP}+\frac{1}{SQ}=\frac{2}{l}.$$

Let the conic have equation

$$\frac{l}{r}=1-e\cos\theta,$$

and let P, Q be the points (r, θ), $(r_1, \pi+\theta)$; then

$$\frac{l}{r}=1-e\cos\theta,$$

and $$\frac{l}{r_1}=1+e\cos\theta;$$

$$\therefore \quad \frac{l}{r}+\frac{l}{r_1}=2,$$

i.e.
$$\frac{1}{r}+\frac{1}{r_1}=\frac{2}{l}.$$

§ **116.** *The chord of the conic,*

$$\frac{l}{r}=1-e\cos\theta,$$

joining the points with vectorial angles α *and* β.

As has been shown in § 22, Note 3, the polar equation of any straight can be written

$$A\cos\theta+B\sin\theta=\frac{1}{r};$$

let the equation of the chord be written

$$p\cos\theta+q\sin\theta=\frac{l}{r}.$$

Then
$$p\cos\alpha+q\sin\alpha=1-e\cos\alpha,$$
and
$$p\cos\beta+q\sin\beta=1-e\cos\beta,$$
i.e.
$$(p+e)\cos\alpha+q\sin\alpha-1=0,$$
and
$$(p+e)\cos\beta+q\sin\beta-1=0;$$

$$\therefore \quad p+e=\frac{\sin\alpha-\sin\beta}{\sin(\alpha-\beta)}=\cos\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2},$$

and
$$q=\frac{\cos\beta-\cos\alpha}{\sin(\alpha-\beta)}=\sin\frac{\alpha+\beta}{2}\sec\frac{\alpha-\beta}{2};$$

$\therefore$ the equation of the chord is

$$\frac{l}{r}=-e\cos\theta+\sec\frac{\alpha-\beta}{2}\left(\cos\theta\cos\frac{\alpha+\beta}{2}+\sin\theta\sin\frac{\alpha+\beta}{2}\right),$$

i.e.
$$\frac{l}{r}=-e\cos\theta+\sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right).$$

§ **117.** *The tangent to the conic*

$$\frac{l}{r} = 1 - e \cos \theta,$$

at the point with vectorial angle α.

The chord joining the points with vectorial angles α, β has equation

$$\frac{l}{r} = -e \cos \theta + \sec \frac{\alpha - \beta}{2} \cos \left(\theta - \frac{\alpha + \beta}{2}\right);$$

therefore the tangent at the point with vectorial angle α has equation

$$\frac{l}{r} = -e \cos \theta + \cos(\theta - \alpha).$$

Example. *P, Q are variable points on the conic,*

$$\frac{l}{r} = 1 - e \cos \theta, \quad \text{......................(i)}$$

with vectorial angles α, β such that $\alpha - \beta = 2\gamma$, where γ is constant; show that the chord PQ touches the conic

$$\frac{l \cos \gamma}{r} = 1 - e \cos \gamma \cos \theta,$$

and that this conic has the same directrix as the conic (i).

The chord PQ has equation

$$\frac{l}{r} = -e \cos \theta + \sec \frac{\alpha - \beta}{2} \cos \left(\theta - \frac{\alpha + \beta}{2}\right)$$

$$= -e \cos \theta + \sec \gamma \cos \left(\theta - \frac{\alpha + \beta}{2}\right)$$

i.e. $$\frac{l \cos \gamma}{r} = -e \cos \gamma \cos \theta + \cos \left(\theta - \frac{\alpha + \beta}{2}\right);$$

this is the tangent at the point with vectorial angle $\frac{\alpha + \beta}{2}$ to the conic

$$\frac{l \cos \gamma}{r} = 1 - e \cos \gamma \cos \theta. \quad \text{...........................(ii)}$$

The length of the perpendicular from the focus to the directrix of (i) is $\frac{l}{e}$; therefore the directrix of (i) has equation

$$\frac{l}{r} = -e \cos \theta.$$

Similarly the directrix of (ii) has equation

$$\frac{l \cos \gamma}{r} = -e \cos \gamma \cos \theta,$$

i.e.

$$\frac{l}{r} = -e \cos \theta.$$

§ **118.** *The normal to the conic,*

$$\frac{l}{r} = 1 - e \cos \theta,$$

at the point P with vectorial angle α.

The polar equation of any straight line can be written

$$A \cos \theta + B \sin \theta = \frac{1}{r};$$

the polar equation of any straight line perpendicular to this is

$$A \cos\left(\theta + \frac{\pi}{2}\right) + B \sin\left(\theta + \frac{\pi}{2}\right) = \frac{1}{r},$$

where k is a constant depending on the position of the line.

The tangent at P has equation

$$\frac{l}{r} = -e \cos \theta + \cos(\theta - \alpha);$$

therefore the equation of any line perpendicular to the tangent is

$$\frac{kl}{r} = -e \cos\left(\theta + \frac{\pi}{2}\right) + \cos\left(\theta + \frac{\pi}{2} - \alpha\right).$$

If the line passes through the point P,

$$k(1-e\cos\alpha)=-e\cos\left(\alpha+\frac{\pi}{2}\right)+\cos\frac{\pi}{2},$$

i.e.
$$k=\frac{e\sin\alpha}{1-e\cos\alpha};$$

$\therefore$ the equation of the normal at P is

$$\frac{e\sin\alpha}{1-e\cos\alpha}\cdot\frac{l}{r}=\sin\theta-\sin(\theta-\alpha),$$

§ **119.** *The chord of contact of tangents from* $P(r_1, \theta_1)$ *to the conic*

$$\frac{l}{r}=1-e\cos\theta.$$

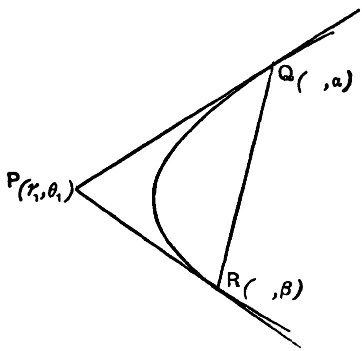

FIG. 68.

Let Q, R (Fig. 68), the extremities of the chord of contact, have vectorial angles α, β; then the chord has equation

$$\frac{l}{r}=-e\cos\theta+\sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right),$$

i.e.
$$\cos\frac{\alpha-\beta}{2}\left(\frac{l}{r}+e\cos\theta\right)=\cos\left(\theta-\frac{\alpha+\beta}{2}\right).$$

Now the tangent at Q has equation

$$\frac{l}{r} = -e\cos\theta + \cos(\theta - \alpha),$$

and P lies on this tangent ; therefore

$$\frac{l}{r_1} = -e\cos\theta_1 + \cos(\theta_1 - \alpha), \qquad \text{(i)}$$

and similarly $$\frac{l}{r_1} = -e\cos\theta_1 + \cos(\theta_1 - \beta). \qquad \text{(ii)}$$

Hence we may write

$$\theta_1 - \alpha = -(\theta_1 - \beta),$$

i.e. $$\theta_1 = \frac{\alpha+\beta}{2},$$

and therefore, from (ii), $$\frac{l}{r_1} = -e\cos\theta_1 + \cos\frac{\alpha-\beta}{2}.$$

The equation of the chord is therefore

$$\left(\frac{l}{r_1} + e\cos\theta_1\right)\left(\frac{l}{r} + e\cos\theta\right) = \cos(\theta - \theta_1).$$

Note. From equations (i) and (ii) we get

$$\theta_1 - \alpha = 2n\pi \pm (\theta_1 - \beta),$$

where n is integral ; the positive sign is irrelevant, and the lower sign gives

$$\theta_1 = n\pi + \frac{\alpha+\beta}{2}.$$

There is no loss of generality in expressing α and β, so that n is zero, i.e. we may write

$$\theta_1 = \frac{\alpha+\beta}{2},$$

so long as we select the corresponding value of $\cos\frac{\alpha-\beta}{2}$, when substituting in the equation of the chord.

Exercises

1. ASB, CSD are two mutually perpendicular focal chords of a conic; show that

$$\frac{1}{AS.SB}+\frac{1}{CS.SD}$$

is constant.

2. Prove that mutually perpendicular focal chords of a rectangular hyperbola are equal in length.

3. Show that the locus of the mid-points of focal chords of the conic

$$r=\frac{l}{1-e\cos\theta},$$

has equation

$$r=l\frac{e\cos\theta}{1-e^2\cos^2\theta}.$$

4. Show that the polar equations of the conics

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1,$$

referred to the origin as pole and the x-axis as initial line are

$$r^2(b^2\cos^2\theta\pm a^2\sin^2\theta)=a^2b^2.$$

5. Show that the polar equation of an ellipse with the centre as pole can be written

$$r^2=\frac{b^2}{1-e^2\cos^2\theta}.$$

6. Show that the polar equation of a hyperbola with the centre as pole can be written

$$r^2=\frac{b^2}{e^2\cos^2\theta-1}.$$

7. Show that the equation

$$\frac{l}{r}=2\sin^2\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$$

represents a parabola with axis perpendicular to the initial line.

8. Transform the equation

$$\frac{l}{r}=2\sin^2\frac{\theta}{2}$$

into Cartesian co-ordinates.

9. Show that the second directrix of the conic

$$\frac{l}{r}=1-e\cos\theta$$

has equation

$$r\cos\theta=\frac{l}{e}\cdot\frac{e^2+1}{e^2-1}.$$

10. Taking the initial line in polar co-ordinates as running positively from left to right, and the vectorial angle measured positively in the counter-clockwise sense, sketch the curves

$$\text{(i) } \frac{l}{r}=1+e\cos\theta, \quad \text{(ii) } \frac{l}{r}=1-e\cos\theta \quad (e<1).$$

Show that the equation of the line joining the points $P_1(r_1, \theta_1)$, $P_2(r_2, \theta_2)$ on the ellipse $\frac{l}{r}=1+e\cos\theta$ is

$$\frac{l}{r}=e\cos\theta+\sec\frac{\theta_2-\theta_1}{2}\cos\left(\theta-\frac{\theta_1+\theta_2}{2}\right).$$

If P_1P_2 meets the major axis of the ellipse in M, and the latus rectum through the pole S in R, show that

$$\left(\frac{l}{SM}\pm e\right)^2+\left(\frac{l}{SR}\right)^2=\sec^2\frac{\theta_2-\theta_1}{2}. \quad \text{(Lond. H.S.C.)}$$

11. Show that the equations of the asymptotes of the hyperbola

$$\frac{l}{r}=1-e\cos\theta$$

can be written

$$b=r\sin\left(\theta-\cos^{-1}\frac{1}{e}\right) \quad \text{and} \quad b=-r\sin\left(\theta+\cos^{-1}\frac{1}{e}\right),$$

where e is the eccentricity and b is the length of the conjugate semi-axis.

12. Show that the equation of the asymptotes of the hyperbola

$$\frac{l}{r}=1-e\cos\theta$$

is

$$\left\{\frac{el}{r}+(e^2-1)\cos\theta\right\}^2=(e^2-1)\sin^2\theta.$$

13. P is an extremity of the latus rectum of the conic

$$\frac{l}{r}=1-e\cos\theta\,;$$

show that the tangent at P makes with the initial line an angle whose tangent is e.

14. P is the point with vectorial angle α on the parabola

$$\frac{l}{r}=2\sin^2\frac{\theta}{2},$$

with focus S; the tangent at P meets the initial line at T; show that $ST=SP=\frac{l}{2}\operatorname{cosec}^2\frac{\alpha}{2}$.

15. Show, using the method of § **116**, that the chord joining the points on the conic,

$$\frac{l}{r}=1-e\cos\theta$$

with vectorial angles $\alpha+\beta$ and $\alpha-\beta$, has equation

$$\frac{l}{r}=-e\cos\theta+\sec\beta\cos(\theta-\alpha)\,;$$

derive the equation of the tangent at the point with vectorial angle α.

16. The tangent at the point P with vectorial angle α on the parabola

$$\frac{l}{r}=2\sin^2\frac{\theta}{2}$$

meets the initial line at Q; show that the equation of the circle passing through P, Q and the pole is

$$\frac{4r}{l}\sin^4\frac{\alpha}{2}=-\cos\theta+\cos(\theta-\alpha).$$

17. P, Q are points with vectorial angles α, β on the conic

$$\frac{l}{r}=1-e\cos\theta\,;$$

TP, TQ are tangents; show that T is the point

$$\left(\frac{l}{\cos\frac{\alpha-\beta}{2}-e\cos\frac{\alpha+\beta}{2}},\ \frac{\alpha+\beta}{2}\right).$$

18. The tangents at the points $(r_1,\ \theta_1)$, $(r_2,\ \theta_1+2\phi)$ on the conic

$$\frac{l}{r}=1-e\cos\theta$$

intersect at the point P; r_3 is the radius vector of P; show that

$$\frac{2}{r_3}=\sec\phi\left(\frac{1}{r_1}+\frac{1}{r_2}-\frac{2}{l}\sin^2\phi\right).$$

19. Show that the locus of the intersection of mutually perpendicular tangents to the conic

$$\frac{l}{r}=1-e\cos\theta$$

has equation

$$\frac{2l}{r}\left(\frac{l}{r}+e\cos\theta\right)=1-e^2.$$

What is this locus when e is unity?

20. P, Q are points on a conic with focus S; TP, TQ are tangents; show that TS bisects angle PSQ.

21. P, Q are points on a parabola with focus S; TP, TQ are tangents; show that $SP \,.\, SQ=ST^2$.

22. Tangents to a parabola are inclined to each other at a constant angle; show that the points of intersection lie on a hyperbola with the same focus as the parabola.

23. Prove that the equations of the tangent and normal to the parabola $2a=r(1+\cos\theta)$ at the point whose angular co-ordinate is α are

$$a=r\cos\tfrac{1}{2}\alpha\cos(\theta-\tfrac{1}{2}\alpha),$$

and

$$a\sin\tfrac{1}{2}\alpha\sec^2\tfrac{1}{2}\alpha=r\sin(\theta-\tfrac{1}{2}\alpha).$$

Prove that the equation of the circle circumscribed about the triangle formed by the tangents at the points whose angular co-ordinates are α, β, γ is

$$r\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma=a\cos(\theta-\tfrac{1}{2}\overline{\alpha+\beta+\gamma}),$$

and that, if the corresponding normals meet at the point r', θ', this is the circle

$$r+r'\cos(\theta-\theta')=0. \qquad \text{(Oxf. H.S.C.)}$$

24. P is a point on the conic

$$\frac{l}{r}=1-e\cos\theta$$

with focus S; P has vectorial angle α and PT is the tangent at P; show that the tangent of the angle SPT is

$$\pm\frac{e\cos\alpha-1}{e\sin\alpha},$$

25. Show that the line

$$\frac{l}{r}=a\cos\theta+b\sin\theta$$

touches the conic

$$\frac{l}{r}=1-e\cos\theta$$

if $(a+e)^2+b^2=1$.

26. Show that tangents at the extremities of a focal chord of a conic intersect on the directrix.

27. P, Q are points with vectorial angles α, β on the parabola

$$\frac{l}{r}=1-\cos\theta\,;$$

show that the angle between the tangents at P and Q is $\dfrac{\alpha-\beta}{2}$.

28. The normal at the point with vectorial angle α on the conic

$$\frac{l}{r}=1-e\cos\theta$$

meets the curve again at the point with vectorial angle β; show that

$$\frac{\cos\dfrac{\beta+\alpha}{2}}{\cos\dfrac{\beta-\alpha}{2}}=e+\frac{1}{e}-\cos\alpha.$$

29. Show that the polar of the point (r_1, θ_1) with respect to the conic

$$\frac{l}{r}=1-e\cos\theta$$

has equation

$$\left(\frac{l}{r}+e\cos\theta\right)\left(\frac{l}{r_1}+e\cos\theta_1\right)=\cos(\theta-\theta_1).$$

30. Three normals are drawn from a point P to the parabola

$$\frac{l}{r}=1-\cos\theta\,;$$

the vectorial angles of P and of the feet of the normals are respectively ϕ, α, β, γ; show that

$$\alpha+\beta+\gamma-2\phi=(2n+1)\pi,$$

where n is integral.

CHAPTER XVIII

GENERAL EQUATION OF THE SECOND DEGREE

§ **120.** *A point, a circle, a pair of straight lines as particular cases of conics.*

(*a*) *A point.*

The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\text{(i)}$$

represents an ellipse with semi-axes of lengths a and b; if a and b are both zero, equation (i) will represent a point.

(*b*) *A circle.*

The equation (i) represents an ellipse with eccentricity $\sqrt{a^2-b^2}/a$ and with directrix distant $a^2/\sqrt{a^2-b^2}$ from the centre.

The equation

$$x^2+y^2=a^2 \quad\text{(ii)}$$

represents a circle and is identical with equation (i) if $b^2=a^2$.

The circle (ii) is therefore a conic with eccentricity zero and directrix at infinity

(*c*) *A pair of straight lines.*

If the y-axis is taken as the directrix of a conic with eccentricity e and if a point on the directrix, say the origin, is taken as focus, the equation of the conic is

$$x^2+y^2=e^2x^2,$$

i.e. $$(e^2-1)x^2-y^2=0,$$

which represents two real and distinct lines, two real and coincident lines or two imaginary lines according as $e \gtreqless 1$.

In each case the two lines pass through the focus; if therefore the focus is a point on a directrix which is at infinity, the two lines will be parallel.

Hence a point, a circle, and a pair of straight lines which are real or imaginary, intersecting, coincident or parallel are particular cases of conics.

§ **121.** *Every equation of the second degree represents a conic.*

The general equation of the second degree is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \text{.............(i)}$$

Turning the axes through angle θ, equation (i) becomes

$$a(x \cos\theta - y \sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta)$$
$$+ 2f(x\sin\theta + y\cos\theta) + c = 0. \quad \text{.............(ii)}$$

In equation (ii) the coefficient of xy is

$$2(b-a)\sin\theta\cos\theta + 2h(\cos^2\theta - \sin^2\theta),$$

and is therefore zero if

$$\tan 2\theta = \frac{2h}{a-b}. \quad \text{.............(iii)}$$

Letting θ have a value satisfying equation (iii), equation (ii) takes the form

$$Ax^2 + By^2 + 2Gx + 2Fy + c = 0. \quad \text{.............(iv)}$$

(a) If $A \neq 0$ and $B \neq 0$, equation (iv) can be written

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - c$$
$$= \lambda, \text{ say.} \quad \text{.............(v)}$$

Changing the origin to the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, equation (v) becomes

$$Ax^2 + By^2 = \lambda, \quad \text{.............(vi)}$$

and therefore represents an ellipse or a hyperbola according as A and B have the same or opposite signs.

(*b*) If $A=0$ and $B \neq 0$, equation (iv) can be written

$$B\left(y+\frac{F}{B}\right)^2=-2Gx+\frac{F^2}{B}-c, \quad \ldots\ldots\ldots\ldots\ldots\text{(vii)}$$

and therefore represents a pair of parallel straight lines if $G=0$. If $G \neq 0$, equation (vii) can be written

$$\left(y+\frac{F}{B}\right)^2=-\frac{2G}{B}\left(x-\frac{F^2}{2BG}+\frac{c}{2G}\right),$$

and therefore represents a parabola.

Similar results are obtained if $A \neq 0$ and $B=0$.

Hence, including as conics the particular cases considered in § **120**, we see that every equation of the second degree represents a conic.

Note. The reader should remember that, after the axes have been rotated through the angle θ, the ellipse or hyperbola has its axes parallel to the axes of co-ordinates; the equation (iii) therefore gives the inclination of the axes of the conic to the original axes of co-ordinates.

§ 122. *Invariants.*

On changing the origin to the point (x_1, y_1) and rotating the axes through the angle θ, the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

becomes

$$\begin{aligned} &a(x\cos\theta-y\sin\theta+x_1)^2 \\ &\quad+2h(x\cos\theta-y\sin\theta+x_1)(x\sin\theta+y\cos\theta+y_1) \\ &\quad+b(x\sin\theta+y\cos\theta+y_1)^2+2g(x\cos\theta-y\sin\theta+x_1) \\ &\quad+2f(x\sin\theta+y\cos\theta+y_1)+c=0, \end{aligned}$$

i.e. becomes

$$a_1x^2+2h_1xy+b_1y^2+2g_1x+2f_1y+c_1=0,$$

where

$$2a_1 = 2(a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)$$
$$= (a+b) + (a-b)\cos 2\theta + 2h\sin 2\theta,$$
$$2b_1 = 2(a\sin^2\theta - 2h\sin\theta\cos\theta + b\cos^2\theta)$$
$$= (a+b) - (a-b)\cos 2\theta - 2h\sin 2\theta,$$

and $2h_1 = -2a\sin\theta\cos\theta + 2h\cos^2\theta - 2h\sin^2\theta + 2b\sin\theta\cos\theta$

$$= -(a-b)\sin 2\theta + 2h\cos 2\theta,$$

and therefore

$$a_1 + b_1 = a + b,$$

and
$$4(a_1b_1 - h_1)^2 = (a+b)^2 - \{(a-b)\cos 2\theta + 2h\sin 2\theta\}^2$$
$$- \{-(a-b)\sin 2\theta + 2h\cos 2\theta\}^2$$
$$= (a+b)^2 - (a-b)^2 - 4h^2$$
$$= 4(ab - h^2),$$

i.e.
$$a_1b_1 - h_1{}^2 = ab - h^2.$$

The expressions $a+b$ and $ab-h^2$ are called invariants.

Note. If the axes are translated but not rotated, the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

becomes an equation of the form

$$ax^2 + 2hxy + by^2 + 2Gx + 2Fy + C = 0;$$

the coefficients of the terms of second degree are unaltered.

§ **123.** *The conic*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{..............(i)}$$

is an ellipse, a parabola or a hyperbola according as

$$h^2 - ab \lesseqgtr 0.$$

The conic is an ellipse if the given equation can be reduced to the form

$$a_1x^2 + b_1y^2 = 1, \quad \text{..................................(ii)}$$

where a_1 and b_1 have the same sign, i.e. if $-a_1b_1$ and therefore $h^2 - ab$ is negative.

Similarly the conic is a hyperbola if $h^2 - ab$ is positive.

The conic is a parabola if the given equation can be reduced to the form

$$b_1y^2+2g_1x=0,$$

i.e. if a_1 and h_1 and therefore h^2-ab is zero.

Note 1. The conic (ii) is a rectangular hyperbola if $a_1+b_1=0$; therefore the condition that equation (i) should represent a rectangular hyperbola is $h^2-ab>0$ and $a+b=0$.

Note 2. We know that equation (i) represents a circle if $a=b$ and $h=0$, and represents two straight lines if

$$abc+2fgh-af^2-bg^2-ch^2=0;$$

the two straight lines are parallel if h^2-ab is zero.

Example. Determine the nature of the following loci:

$$\text{(i)}\ 2x^2-3xy-2y^2+x-2y-1=0;$$
$$\text{(ii)}\ 3x^2-2xy+3y^2+2x-6y+2=0;$$
$$\text{(iii)}\ 4x^2-4xy+y^2+6x-3y+2=0.$$

(i) If $a=2$ and $f=-1$,
$b=-2$ $\quad g=\frac{1}{2}$,
$c=-1$ $\quad h=-\frac{3}{2}$,

$$abc+2fgh-af^2-bg^2-ch^2=4+\tfrac{3}{2}-2+\tfrac{1}{2}+\tfrac{9}{4}\neq 0,$$
$$h^2-ab=\tfrac{9}{4}+4>0,$$

and $$a+b=0;$$

therefore equation (i) represents a rectangular hyperbola.

(ii) If $a=3$ and $f=-3$,
$b=3$ $\quad g=1$,
$c=2$ $\quad h=-1$,

$$abc+2fgh-af^2-bg^2-ch^2=18+6-27-3-2\neq 0,$$

and $$h^2-ab=1-9<0;$$

therefore equation (ii) represents an ellipse.

(iii) If $a=4$ and $f=-\frac{3}{2}$,
$b=1$ $\quad g=3$,
$c=2$ $\quad h=-2$,

$$abc+2fgh-af^2-bg^2-ch^2=8+18-9-9-8=0,$$

and $$h^2-ab=4-4=0\,;$$

therefore equation (iii) represents two parallel straight lines.

§ **124.** *To find the centre of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \quad \text{..............(i)}$$

Changing the origin to the point (x_1, y_1), equation (i) becomes

$$ax^2+2hxy+by^2+2(ax_1+hy_1+g)x+2f(hx_1+by_1+f)$$
$$+ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c=0. \quad \text{..............(ii)}$$

If we choose x_1, y_1 so that

$$ax_1+hy_1+g=0, \quad hx_1+by_1+f=0, \quad \text{...........(iii)}$$

equation (ii) is

$$ax^2+2hxy+by^2+c_1=0, \quad \text{.................(iv)}$$

where $$c_1=ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c.$$

Equation (iv) represents a conic symmetrical about the origin, i.e. the origin is the centre.

Therefore the centre of the conic (i) has co-ordinates given by equations (iii), i.e. the centre is the point

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right).$$

Note 1. If $h^2-ab=0$, the curve is a parabola and the centre is at infinity. If $h^2-ab=0$ and $hf-bg=0$, the centre is indeterminate; in this case the locus is two parallel straight lines.

Note 2. If $ax_1+hy_1+g=0$ and $hx_1+by_1+f=0$,

$$ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c,$$

which
$$=x_1(ax_1+hy_1+g)+y_1(hx_1+by_1+f)+gx_1+fy_1+c$$
$$=gx_1+fy_1+c$$
$$=g\,\frac{hf-bg}{ab-h^2}+f\,\frac{gh-af}{ab-h^2}+c$$
$$=\frac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}.$$
$$=\frac{\Delta}{ab-h^2}.$$

Therefore the equation of the conic (1) when referred to its centre as origin is

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0,$$

which is of the form

$$a_1x^2 + 2h_1xy + b_1y^2 = 1.$$

The expression denoted by Δ has already appeared in discussing the general equation of the second degree; $\Delta = 0$ is the condition that the equation should represent two straight lines.

Example. *Find the centre of the conic*

$$7x^2 + 48xy - 7y^2 + 20x - 110y - 50 = 0$$

and the equation of the conic referred to parallel axes through its centre.

At the centre,

$$7x + 24y + 10 = 0,$$

and $$24x - 7y - 55 = 0,$$

i.e. $$x = \frac{-1320 + 70}{-49 - 576} = 2,$$

and $$y = \frac{240 + 385}{-625} = -1,$$

i.e. the centre is the point $(2, -1)$, and the equation referred to parallel axes through the centre is

$$7x^2 + 48xy - 7y^2 + c = 0,$$

where $$c = 28 - 96 - 7 + 40 + 110 - 50 = 25,$$

i.e. the equation is

$$7x^2 + 48xy - 7y^2 + 25 = 0.$$

§ 125. *To trace a central conic.*

The method is illustrated by the following examples.

Example 1. *Trace the conic*

$$11x^2 + 4xy + 14y^2 - 4x - 28y - 16 = 0.$$

Since $4 - 11 \times 14 < 0$ the curve is an ellipse.

At the centre $11x+2y-2=0,$

and $2x+14y-14=0;$

$\therefore$ the centre is the point (0, 1).

The equation referred to parallel axes through the centre is

$$11x^2+4xy+14y^2+c=0,$$

where $c=14-28-16=-30,$

i.e. is $11x^2+4xy+14y^2-30=0.$ ....(i)

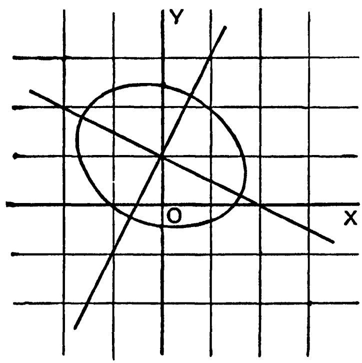

FIG. 69.

If an axis of the ellipse is inclined at angle θ to the x-axis

$$\tan 2\theta=\frac{4}{11-14},$$

i.e. $$\frac{2\tan\theta}{1-\tan^2\theta}=-\frac{4}{3},$$

i.e. $$2\tan^2\theta-3\tan\theta-2=0,$$

i.e. $$\tan\theta=2 \text{ or } -\tfrac{1}{2}.$$

Equation (i) expressed in polar co-ordinates is

$$r^2(11\cos^2\theta+4\cos\theta\sin\theta+14\sin^2\theta)=30(\cos^2\theta+\sin^2\theta),$$

i.e. $$r^2=30\,\frac{1+\tan^2\theta}{11+4\tan\theta+14\tan^2\theta};$$

$\therefore$ when $\tan\theta=2$, $\quad r^2=30\times\frac{5}{75}=2$,

and when $\tan\theta=-\frac{1}{2}$, $r^2=30\times\dfrac{\frac{5}{4}}{\frac{25}{2}}=3$;

$\therefore$ the lengths of the semi-axes are $\sqrt{3}$ and $\sqrt{2}$.

The axes are the lines passing through the point (0, 1) and having gradients $-\frac{1}{2}$ and 2; therefore the equations of the axes are

$$x+2y-2=0, \quad 2x-y+1=0.$$

The conic is therefore as indicated in Fig. 69.

Example 2. *Trace the conic*

$$3x^2-10xy+3y^2+14x-2y+3=0.$$

Since $25-3\times3>0$ the curve is a hyperbola.

At the centre $\qquad 3x-5y+7=0,$

and $\qquad -5x+3y-1=0$;

$\therefore$ the centre is the point (1, 2).

The equation referred to parallel axes through the centre is

$$3x^2-10xy+3y^2+c=0,$$

where $\qquad c=3-20+12+14-4+3=8,$

i.e. is $\qquad 3x^2-10xy+3y^2+8=0. \qquad \text{.................(i)}$

If an axis of the hyperbola is inclined at angle θ to the x-axis,

$$\tan 2\theta=\frac{-10}{3-3},$$

i.e. $\qquad \tan\theta=1 \text{ or } -1.$

Equation (i) expressed in polar coordinates is

$$r^2=-8\frac{1+\tan^2\theta}{3-10\tan\theta+3\tan^2\theta};$$

$\therefore$ when $\tan\theta=1$, $\quad r^2=-8\times\dfrac{2}{-4}=4$,

and when $\tan\theta=-1$, $\quad r^2=-8\times\frac{2}{16}=-1$;

$\therefore$ the transverse and conjugate semi-axes have lengths 2 and 1 respectively.

The equations of these axes are

$$x-y+1=0, \quad x+y-3=0.$$

The conic is therefore as indicated in Fig. 70.

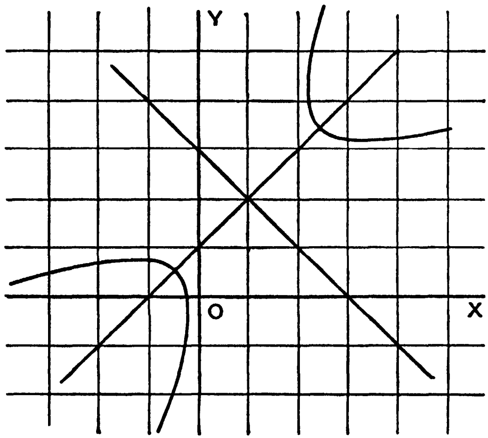

FIG. 70.

Note. When sketching a conic the reader should check his calculations by determining the points of intersection of the conic and the axes of co-ordinates.

§ **126.** *The asymptotes of the conic*

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{..............(i)}$$

have equation

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{\Delta}{ab-h^2},$$

where $\quad \Delta=abc+2fgh-af^2-bg^2-ch^2.$

When a hyperbola has its equation written in the form

$$\frac{x^2}{a^2}-\frac{y^2}{\beta^2}=1, \quad \text{..............................(ii)}$$

the asymptotes are represented by the equation

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

i.e. the equations of the curve and of its asymptotes differ only in the constant term. If the origin is changed and the axes are rotated and the equations (ii) and (iii) are transformed accordingly, the new equations will differ only in the constant term. Therefore the asymptotes of the conic (i) have equation

$$ax^2+2hxy+by^2+2gx+2fy+c=\lambda, \quad \ldots\ldots\ldots\ldots\ldots\text{(iv)}$$

where λ is a constant.

But equation (iv) does not represent two straight lines unless

$$ab(c-\lambda)+2fgh-af^2-bg^2-(c-\lambda)h^2=0,$$

i.e. unless
$$\lambda=\frac{\Delta}{ab-h^2};$$

therefore the asymptotes have equation

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{\Delta}{ab-h^2}. \quad \ldots\ldots\ldots\ldots\ldots\text{(v)}$$

Note 1. The asymptotes of the conic

$$(ax+by+c)(a_1x+b_1y+c_1)=\text{a constant},$$

have equation

$$(ax+by+c)(a_1x+b_1y+c_1)=0,$$

for the latter equation represents two straight lines and differs from the former only in the constant term.

Note 2. Since the equations of a hyperbola, its asymptotes and the conjugate hyperbola can be written

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=0, \quad \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

it follows from (v) that the hyperbola conjugate to the hyperbola (i) has equation

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{2\Delta}{ab-h^2}.$$

§ **127.** *To trace a parabola.*

Two methods are illustrated by the following examples.

Example 1. *Trace the parabola*

$$x^2-4xy+4y^2-6x-8y+5=0.$$

The equation can be written

$$(x-2y)^2-6x-8y+5=0.$$

Take $x-2y=0$ as a new x-axis, i.e. turn the axes through angle θ where $\tan\theta=\frac{1}{2}$; the new equation is

$$5y^2-6\frac{2x-y}{\sqrt{5}}-8\frac{x+2y}{\sqrt{5}}+5=0,$$

i.e. $$5\sqrt{5}y^2-20x-10y+5\sqrt{5}=0,$$

i.e. $$y^2-\frac{4}{\sqrt{5}}x-\frac{2}{\sqrt{5}}y+1=0,$$

i.e. $$\left(y-\frac{1}{\sqrt{5}}\right)^2=\frac{4}{\sqrt{5}}\left(x-\frac{1}{\sqrt{5}}\right);$$

$\therefore$ the parabola has vertex $\left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ referred to the new axes and has latus rectum of length $\frac{4}{\sqrt{5}}$.

The curve is as indicated in Fig. 71.

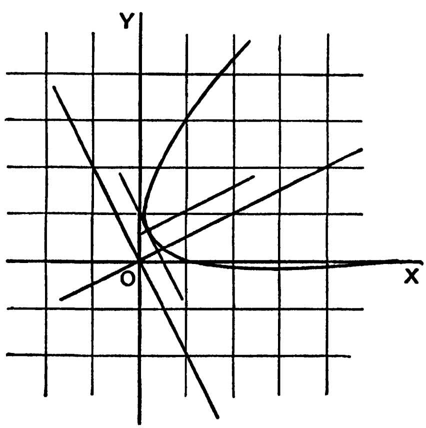

FIG. 71.

Example 2. *Trace the parabola*

$$4x^2+4xy+y^2+2x+6y+3=0.$$

The equation can be written

$$(2x+y+\lambda)^2=(4\lambda-2)x+(2\lambda-6)y+\lambda^2-3. \quad \text{......(i)}$$

The lines

$$2x+y+\lambda=0,$$

and $\quad (4\lambda-2)x+(2\lambda-6)y+\lambda^2-3=0,$

are mutually perpendicular if

$$-2\times\left(-\frac{2\lambda-1}{\lambda-3}\right)=-1,$$

i.e. if $\qquad \lambda=1.$

With this value of λ, equation (i) is

$$(2x+y+1)^2=2(x-2y-1),$$

i.e. $$\left(\frac{2x+y+1}{\sqrt{5}}\right)^2=\frac{2}{\sqrt{5}}\left(\frac{x-2y-1}{\sqrt{5}}\right),$$

which represents a parabola with latus rectum of length $\dfrac{2}{\sqrt{5}}$, with axis

$$2x+y+1=0$$

and with tangent at the vertex

$$x-2y-1=0;$$

the vertex is therefore the point $(-\frac{1}{5}, -\frac{3}{5})$.

The parabola is as indicated in Fig. 72.

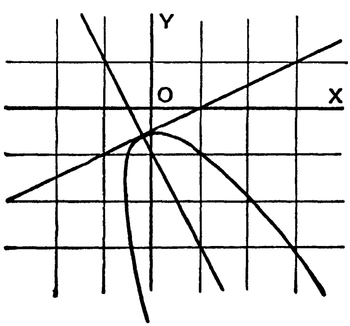

FIG. 72.

Exercises

Determine the nature of the loci:

1. $11x^2 - 6xy + 19y^2 + 6x + 2y - 1 = 0.$
2. $x^2 + 16xy - 11y^2 - 6x + 12y - 6 = 0.$
3. $9x^2 - 12xy + 4y^2 + 4x - 7y + 1 = 0.$
4. $3x^2 + 3y^2 + 4x - 2y - 1 = 0.$
5. $6x^2 - xy - y^2 - x + 3y - 2 = 0.$
6. $9x^2 - 12xy + 4y^2 - 3x + 2y - 2 = 0.$
7. $3x^2 + 8xy - 3y^2 + 4x + 2y + 3 = 0.$

8. P is a variable point whose co-ordinates are given by equations

$$x = 2t^2 - 1, \quad y = t^2 - 2t + 3;$$

find the equation of the locus of P and show that the locus is parabola.

9. P is a variable point whose co-ordinates are given by equations

$$x = at^2 + bt + c, \quad y = dt^2 + et + f;$$

show that the locus of P is a parabola.

10. Find the centre of the conic

$$x^2 - 2xy + 3y^2 - 10x + 22y + 30 = 0,$$

and the equation referred to parallel axes through the centre.

11. Find the centre of the conic

$$3x^2 + 2xy + 4y^2 - 10x - 18y + 28 = 0,$$

and the equation referred to parallel axes through the centre.

Find the lengths and equations of the axes of the conics:

12. $11x^2 - 24xy + 4y^2 + 20 = 0,$
13. $8x^2 + 4xy + 5y^2 - 8x + 16y - 16 = 0,$
14. $6x^2 + 4xy + 9y^2 - 24x - 58y + 89 = 0,$
15. $23x^2 + 72xy + 2y^2 - 20x - 140y = 0,$
16. $x^2 - 4xy - 2y^2 + 6x + 12y - 21 = 0.$

17. Show that the lines passing through the origin and the sections of the conic

$$ax^2 + 2hxy + by^2 = 1$$

with the circle

$$x^2+y^2=r^2,$$

have equation

$$\left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0\,;$$

deduce that, if r is the length of a semi-axis of the conic,

$$h^2=\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right),$$

and that the axis has equation

$$\left(a-\frac{1}{r^2}\right)x+hy=0.$$

18. Using the result of Exercise **17**, show that if r is the length of a semi-axis of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

$$(ab-h^2)^3r^4+\Delta(a+b)(ab-h^2)r^2+\Delta^2=0,$$

where $\Delta=abc+2fgh-af^2-bg^2-ch^2$.

Find the equation of the asymptotes of each of the following hyperbolas :

19. $2x^2-12xy-7y^2-4x-8y-4=0,$

20. $2x^2+xy-y^2-x+2y-3=0,$

21. $x^2-y^2+x+y-1=0.$

22. A hyperbola has equation

$$f(x,\,y)=0,$$

where $f(x,\,y)\equiv ax^2+2hxy+by^2+2gx+2fy+c$; the centre is the point $(x_1,\,y_1)$; show that the asymptotes have equation

$$f(x,\,y)=f(x_1,\,y_1).$$

23. Find the equations of (i) the asymptotes, (ii) the axes of the conic given by the equation

$$x^2+y^2-6xy+10x+2y=0. \qquad \text{(Camb. H.S.C.)}$$

24. Find the equation of the hyperbola having the same asymptotes as the hyperbola

$$2x^2-3xy-2y^2+2x+11y-13=0,$$

and passing through the point (3, 2).

25. A hyperbola has equation

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0\,;$$

show that the equation

$$f(x, y) = f(x_1, y_1)$$

represents a hyperbola having the same asymptotes and passing through the point (x_1, y_1).

26. Show that the equation of the asymptotes of the hyperbola

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

can be written

$$b\xi^2 - 2h\xi\eta + a\eta^2 = c,$$

where $$\xi \equiv ax + hy + g, \quad \eta \equiv hx + by + f.$$

27. Find the equation of the hyperbola conjugate to the hyperbola

$$x^2 - 6xy + y^2 + 4x + 4y - 2 = 0.$$

Trace the parabolas :

28. $x^2 - 4xy + 4y^2 - 2x - 16y + 25 = 0,$

29. $9x^2 - 24xy + 16y^2 - 20x - 140y = 0,$

30. $4x^2 - 4xy + y^2 - 3x - 6y + 3 = 0.$

31. Find λ so that the lines whose equations are

$$8x + 15y + \lambda = 0,$$

and $$(221 + 16\lambda)x + (30\lambda - 1428)y + \lambda^2 - 2023 = 0$$

may be at right angles to one another.

Trace the parabola

$$64x^2 + 240xy + 225y^2 - 221x + 1428y + 2023 = 0.$$

(Lond. H.S.C.)

32. Show that the latus rectum of the parabola

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

has length

$$\frac{2(a^{\frac{1}{2}}f - b^{\frac{1}{2}}g)}{(a+b)^{\frac{3}{2}}}.$$

REVISION EXERCISES

A. On the Parabola

Find the equations of the parabolas with the following foci and directrices:

1. $(-1, 2)$, $x+2=0$. **2.** $(3, 1)$, $x=7$.

3. $(1, -2)$, $x+y-2=0$.

Find the vertices and foci of the following parabolas:

4. $y^2=2(x-1)$. **5.** $y^2-8x-2y+25=0$.

6. $y^2+x-2y+2=0$.

7. Show that the parabola, $y^2=4ax$, is the locus of a point whose distance from the line

$$x=3a$$

is equal to the length of the tangent from the point to the circle

$$x^2+y^2-10ax+9a^2=0.$$

8. Show that the parabola, $y^2=4ax$, is the locus of the centre of a circle touching the y-axis and the circle

$$x^2+y^2-2ax=0.$$

9. A variable circle passes through the point $(2a, 0)$ and touches the y-axis; show that the locus of its centre is a parabola and determine its vertex.

10. A circle of radius a touches the x- and y-axes; a point P moves so that its abscissa is equal in length to the tangent from P to the circle; show that the locus of P is a parabola.

11. A is the fixed point $(a, 0)$; B is a variable point on the y-axis; on AB and on the side remote from the origin O, $\triangle ABP$ is drawn similar to $\triangle AOB$; show that the locus of P is the parabola, $y^2=4ax$.

12. Show that the common chord of the parabolas

$$y^2=4ax, \quad y^2=2a(x+a)$$

is the latus rectum of the former.

13. A, B, C are points on the parabola, $y^2 = 4ax$, with ordinates y_1, y_2, y_3; O is the origin; show that

$$\text{(i) } \triangle OAB = \frac{y_1 y_2}{8a}(y_1 - y_2);$$

$$\text{(ii) } \triangle ABC = \frac{1}{8a}(y_1 - y_2)(y_2 - y_3)(y_3 - y_1).$$

14. The parabola, $y^2 = 4ax$, divides the plane into two parts one of which contains the focus; show that throughout this part

$$y^2 - 4ax$$

is negative, while throughout the other part it is positive.

15. Show that the equation

$$y = ax^2 + bx + c$$

represents a parabola, and find the co-ordinates of its vertex and the length of its latus rectum.

16. P is the point (x_1, y_1) and PAB is a line inclined at angle θ to the axis of the parabola

$$y^2 = 4ax,$$

and cutting the parabola at A and B; show that

$$PA \, . \, PB = \frac{y_1^2 - 4ax_1}{\sin^2\theta}.$$

17. Two variable chords AB, CD of a parabola, each drawn in a fixed direction, intersect at P; prove that

$$PA \, . \, PB : PC \, . \, PD$$

is constant.

18. QE, QF, the tangents at E and F, are parallel to AB, CD in Exercise 17, and S is the focus; prove that

$$\frac{PA \, . \, PB}{PC \, . \, PD} = \frac{QE^2}{QF^2} = \frac{SE}{SF}.$$

19. Show that the focus of the parabola

$$x^2 + y^2 = (x \cos \alpha + y \sin \alpha - p)^2$$

is at the origin. What is the length of the latus rectum?

20. Prove (do not merely verify) that the equation of the circle through the points $(p, 0)$, $(q, 0)$, $(0, r)$ is

$$r(x^2 + y^2) - r(p + q)x - (r^2 + pq)y + pqr = 0.$$

A circle passes through a fixed point, and the chord cut off from it by a given line is of constant length. Prove that the locus of its centre is a parabola. (Oxf. and Camb. H.S.C.)

21. P, Q are the points t_1, t_2 on the parabola, $y^2=4ax$, with focus S; find the area of $\triangle SPQ$, and hence show that PQ passes through S if $t_1t_2=-1$.

22. A circle has as diameter a focal chord of the parabola, $y^2=4ax$; the circle cuts the axes of co-ordinates at the points $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$, $(0, y_2)$; show that

$$x_1x_2=y_1y_2=-3a^2.$$

23. The circle

$$x^2+y^2+2gx+2ay=0$$

cuts the parabola

$$y^2=ax$$

at the origin and three points whose ordinates are y_1, y_2, y_3; show that

$$y_1y_2y_3=-2a^3.$$

24. A circle cuts a parabola at A, B, C, D; the tangents to the parabola at A and B meet at T, the tangents at C and D meet at V; show that the axis bisects TV.

25. A is a fixed point on a parabola; a variable circle passing through A meets the parabola again at B, C, D; G is the centroid of $\triangle BCD$; P divides AG in the ratio 3 : 1; show that the locus of P is the axis of the parabola.

26. BC is a variable focal chord of the parabola, $y^2=4ax$; the circle passing through B, C and the vertex A meets the parabola again at D; AC, BD intersect at P; show that the locus of P has equation

$$ay^2+4x^2(2x-a)=0.$$

27. A circle touches a parabola at P and meets the curve again at Q and R; show that the axis bisects the line joining P to the mid-point of the chord QR.

28. The points (x_1, y_1), (x_2, y_2) are the extremities of a focal chord of the parabola, $y^2=4ax$; show that

$$x_1x_2=a^2, \quad y_1y_2=-4a^2.$$

29. Find the length of the chord of the parabola, $y^2=4ax$, passing through the point $(a\tan^2\theta, 2a\tan\theta)$ and inclined at angle ϕ to the axis of the parabola.

30. Show that the line

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

meets the parabola, $y^2=4ax$, where

$$r^2\sin^2\theta+2r(y_1\sin\theta-2a\cos\theta)+y_1{}^2-4ax_1=0,$$

and deduce that if parallel chords are divided so that the product of their segments is constant the locus of the points of section is a parabola congruent with the given one.

31. PT is the tangent at P, a variable point on the parabola, $y^2=4ax$; T lies on the tangent at the vertex; show that the locus of the mid-point of PT is the parabola

$$2y^2=9ax.$$

32. The tangent at P, a variable point on the parabola, $y^2=4ax$, meets the parabola, $y^2=-4ax$, at Q and R; show that the locus of the mid-point of QR is the parabola

$$3y^2+4ax=0.$$

33. PQ is a focal chord of the parabola, $y^2=4ax$; QT is the tangent at Q and TP is parallel to the axis; show that the locus of T has equation

$$y^2(x+2a)+4a^3=0.$$

34. P is a variable point on the parabola, $y^2=4ax$, with focus S and vertex A; PS meets the curve again at Q and the tangent at Q meets PA at T; show that the locus of T has equation

$$4x^3+(2x+a)y^2=0.$$

35. PQ is a chord of a parabola; T is any point on the tangent at P; the line passing through T and parallel to the axis meets PQ at N; show that the parabola divides TN in the ratio $PN:NQ$.

Find the conditions that the following lines should touch the parabola, $y^2=4ax$:

36. $px+qy=1$. **37.** $Ax+By+C=0$. **38.** $\dfrac{x-x_1}{\cos\theta}=\dfrac{y-y_1}{\sin\theta}=r$.

39. Show that the line

$$y=mx+\frac{a}{m}$$

touches the circle

$$x^2+y^2=r^2,$$

if
$$m^4+m^2-\frac{a^2}{r^2}=0\,;$$

hence find the tangents common to the parabola, $y^2=4ax$, and the circle

$$20x^2+20y^2=a^2.$$

40. Find the equations of the tangents common to the parabola, $y^2=4ax$, and the circle

$$2x^2+2y^2=a^2.$$

41. Find the locus of the foot of the perpendicular drawn from the focus of a parabola to a tangent to the curve.

If S is the focus of the parabola and SK the perpendicular to the tangent at P, prove that SK^2 is proportional to SP.

(Jt. Matric. Bd. H.C.)

42. A variable tangent to the parabola, $y^2=4ax$, meets the parabola

$$y^2=4bx$$

at P and Q and cuts the x-axis at R; show that $PR:RQ$ is constant.

43. A, B, C are points on a parabola; the tangents at A, B, C intersect in pairs at D, E, F; show that

$$\triangle ABC=2\triangle DEF.$$

44. TP, TQ are tangents to the parabola, $y^2=4ax$; angle $PTQ=\theta$; show that T lies on the curve

$$y^2-4ax=(x+a)^2\tan^2\theta,$$

and deduce that mutually perpendicular tangents intersect on the directrix.

45. P is a variable point on the parabola, $y^2=4ax$, with focus S; the chord through the vertex and parallel to SP meets the parabola at Q; TP, TQ are tangents; show that the locus of T has equation

$$(x+a)y^2=a(2x+a)^2.$$

46. The circle having as diameter MP, the ordinate of a point P on a parabola, meets the curve again at Q; TP, TQ are tangents to the parabola; NT, the ordinate of T, meets the parabola at R; show that $MQ=NR$.

47. A circle passing through the vertex of a parabola meets the latter again at P, Q, R; the tangents at Q and R meet at T; show that PT is trisected by the axis of the parabola.

48. A circle cuts the parabola, $y^2=4ax$, at the points P, Q, R, S; show that the intersections of the tangents at P, Q, R, S are such that the sum of their ordinates is zero.

49. A circle passing through the vertex of the parabola, $y^2=4ax$, meets the curve again at A, B, C; the tangents at A and B meet the tangent at C at the points P and Q; show that the mid-point of PQ lies on the parabola

$$2y^2+ax=0.$$

50. Show that the normals to the parabola

$$y^2=2x$$

at the extremities of the chord

$$2x-3y+2=0$$

intersect on the parabola.

51. Find the equations of the tangent and normal at the point (x_1, y_1) on the curve $y=2\sqrt{(ax)}$.

Prove that the two tangents to this curve which pass through the point $(-a, k)$ are at right angles. (Cent. Welsh Bd. H.S.C.)

52. The normal at P, a point on the parabola $y^2=4ax$, meets the axis at Q; show that the locus of the mid-point of PQ has equation

$$y^2=a(x-a).$$

53. A, B, C, D are four concyclic points on a parabola; AB is a focal chord; AC is the normal at A; show that the axis divides DB in the ratio 3 : 1.

54. P is a variable point on the parabola, $y^2=4ax$; the chord PQ is the normal at P; TP, TQ are tangents; show that the locus of T has equation

$$(x+2a)y^2+4a^3=0.$$

55. The chord PQ is the normal at P to the parabola, $y^2=4ax$; PT is the tangent at P and QT is parallel to the axis; show that the locus of T has equation

$$3y^2(x+4a)+4a(x-2a)^2=0.$$

56. The normal at P, a point on the parabola $y^2=4ax$, meets the tangent at the vertex at Q; show that the locus of the mid-point of PQ has equation

$$ay^2=2x(x+2a)^2.$$

57. A chord PQ of the parabola, $y^2=4ax$, subtends a right angle at the vertex; show that the normals at P and Q intersect on the parabola

$$y^2=16a(x-6a).$$

58. Normals to the parabola, $y^2=4ax$, from a point whose abscissa is $3a$, meet the parabola at points whose abscissae are x_1, x_2, x_3; show that

$$x_1+x_2+x_3=2a.$$

59. Show that two of the three normals from the point (14, 16) to the parabola

$$y^2=4x$$

coincide.

60. Find the two distinct normals from the point $(\frac{5}{4}, -\frac{1}{2})$ to the parabola

$$y^2=x.$$

61. Q is the foot of the perpendicular from $P(x_1, y_1)$ to the polar of P with respect to the parabola, $y^2=4ax$; show that Q lies on the circle

$$x^2+y^2-y_1y-x_1{}^2=0.$$

62. A variable chord PQ of the parabola, $y^2=4ax$, subtends a right angle at the vertex; find the locus of the pole of PQ with respect to the circle

$$x^2+y^2=a^2.$$

63. The polar of P with respect to the parabola, $y^2=4ax$, touches the parabola

$$y^2+4ax=0\ ;$$

show that P lies on the latter parabola.

64. Tangents are drawn to the circle $x^2+y^2=a^2$; show that the locus of their poles with respect to the parabola, $y^2=4ax$, has equation

$$y^2=4(x^2-a^2).$$

65. Show that the points (α, β), $\left(\dfrac{\beta^2}{2a}-\alpha, \beta\right)$ are conjugate with respect to the parabola, $y^2=4ax$

66. Show that the lines

$$l_1x+m_1y+n_1=0, \quad l_2x+m_2y+n_2=0$$

are conjugate with respect to the parabola, $y^2=4ax$, if

$$l_1n_2+l_2n_1=2am_1m_2.$$

67. P, Q are any two points on a parabola; TP, TQ are tangents; TP meets the diameter through Q at R; show that $RT = TP$.

68. The polar of a variable point P with respect to a parabola is parallel to the line joining P to a fixed point Q; show that the locus of P is a parabola passing through Q.

69. A line through $P(x_1, y_1)$, a fixed point on the parabola

$$y^2 = 8ax,$$

meets the parabola $y^2 = 4ax$ at Q and R; show that the locus of the mid-point of QR is the parabola

$$(2y - y_1)^2 = 8ax.$$

70. The point (x_1, y_1) is the pole of a chord of the parabola $y^2 = 4ax$; show that the mid-point of the chord is the point $\left(\dfrac{y_1{}^2 - 2ax_1}{2a}, y_1\right)$.

71. The mid-point of a chord PQ of the parabola, $y^2 = 4ax$, lies on the parabola

$$y^2 = 2a(x - a);$$

show that the pole of PQ with respect to the former parabola lies on the directrix of that parabola.

72. The mid-point of a chord PQ of the parabola, $y^2 = 4ax$, lies on the parabola

$$y^2 = ax;$$

show that the pole of PQ with respect to the former parabola lies on the parabola

$$y^2 + 2ax = 0.$$

73. The mid-point of a chord PQ of the parabola, $y^2 = 4ax$, lies on the line

$$x = y;$$

show that the locus of the pole of PQ with respect to the parabola has equation

$$y^2 = 2a(x + y).$$

74. P, Q are points on the parabola, $y^2 = 4ax$; NP, NQ are the normals at P, Q; N lies on the line

$$y + a = 0;$$

show that the pole of the chord PQ lies on the conic

$$xy = a^2.$$

75. P, Q are points on the parabola, $y^2=4ax$; NP, NQ are the normals at P, Q; N lies on the line

$$x=y;$$

show that the locus of the pole of the chord PQ has equation

$$y(x+y)=a(x-2a).$$

76. Obtain, in a simple symmetrical form, the equation of the line joining the points $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$.

P, Q, R are three points on a parabola. The diameters through P, Q meet QR, RP at D, E respectively. Prove that the tangents at P, Q intersect at the middle point of DE. (Oxf. and Camb. H.S.C.)

77. Prove that the straight line midway between a point and its polar with respect to a parabola is a tangent to the parabola.

Having given the axis of a parabola and two points on the parabola which are not reflexions of each other in the axis, show how to construct the vertex and focus of the parabola. (Oxf. H.S.C.)

78. If the tangents to a parabola at the points P and Q meet at the point T, prove that the straight line joining T to the middle point of PQ is parallel to the axis of the parabola.

If two tangents to a parabola and their points of contact are given, obtain a construction for the focus and directrix of the parabola. (Oxf. H.S.C.)

B. On the Ellipse

79. Find the equation of the ellipse with focus $(1, 0)$, directrix $x=4$ and eccentricity $\frac{2}{3}$.

80. Find the directrices and eccentricity of the ellipse

$$3x^2+4y^2-12x-16y+16=0.$$

81. Show that in any ellipse the distance of a focus from an extremity of the minor axis is equal to the length of the semi-major axis.

82. Given that S, S' are the points $(\pm ae, 0)$ and that

$$SP+S'P=2a;$$

show that the locus of P has equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $b^2=a^2(1-e^2)$.

83. The lines

$$bx + aty - ab = 0, \quad btx - ay + abt = 0,$$

where t is variable, meet at P; show that the locus of P is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

84. AOA', BOB' are the major and minor axes of an ellipse with focus S; show that $A'S \, . \, SA = OB^2$.

85. P is any point on an ellipse with major axis AA' and focus S; the chords PA, PA' meet the directrix at Q, Q'; show that $QS \perp Q'S$.

86. A, B, C are the points α, β, γ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ;$$

show that the area of triangle ABC is

$$2ab \sin \frac{\beta - \gamma}{2} \sin \frac{\gamma - \alpha}{2} \sin \frac{\alpha - \beta}{2}.$$

87. P, Q are the points θ, 3θ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ;$$

show that the locus of the mid-point of the chord PQ has equation

$$\frac{x^2}{a^2} = \left(2\,\frac{x^2}{a^2} + 2\,\frac{y^2}{b^2} - 1\right)^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right).$$

88. A, B, C are three points in order on a straight line; a variable circle touches the line at C, and the other tangents from A and B to the circle intersect each other at P; show that the locus of P is an ellipse and find its equation, centre and eccentricity when A, B, C are the points $(-2, 0)$, the origin and $(1, 0)$.

89. A circle passing through the point $(0, b)$ meets the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

again at P, Q, R; the gradients of PQ, QR, RP are m_1, m_2, m_3; show that

$$b^2(m_1 + m_2 + m_3) = a^2 m_1 m_2 m_3.$$

90. If (h, k) is the middle point of a chord of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and $2u \equiv x_1 - x_2$, $2v = y_1 - y_2$, where (x_1, y_1), (x_2, y_2) are the extremities of the chord, prove that

$$\frac{h^2+u^2}{a^2}+\frac{k^2+v^2}{b^2}-1=0, \text{ and } \frac{hu}{a^2}+\frac{kv}{b^2}=0.$$

Hence prove that the length of the chord is

$$2ab\sqrt{\left(\frac{h^2}{a^4}+\frac{k^2}{b^4}\right)\left\{\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)^{-1}-1\right\}}. \qquad \text{(Oxf. H.S.C.)}$$

91. Circles are drawn with the two foci of an ellipse as centres to pass through a point on the curve. Show that the common tangents to these circles touch the auxiliary circle, and that their points of contact with this circle are on the common chord of the original circles. (Camb. H.S.C.)

92. CP, CQ are perpendicular semi-diameters of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;$$

show that PQ touches the circle

$$(a^2+b^2)(x^2+y^2)=a^2b^2.$$

93. AB, CD two variable chords of an ellipse are drawn in fixed directions ; AB, CD meet at P ; prove that

$$PA \,.\, PB : PC \,.\, PD$$

is constant.

94. Show that the line

$$\frac{(x-h)\cos\theta}{a}+\frac{(y-k)\sin\theta}{b}=1,$$

where θ is variable, is a tangent to a fixed ellipse.

95. From a point P on a circle a perpendicular PN is drawn to a fixed diameter HK of the circle. If a point Q divides PN in the ratio 2 : 3, prove that the locus of Q is an ellipse, and find its eccentricity.

Prove also that the tangent to the circle at P and the tangent to the ellipse at Q intersect on HK produced. (Jt. Matric. H.S.C.)

96. P, Q are the points 3θ, θ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;$$

show that the tangents at P, Q intersect on the curve

$$y^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=4b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

97. P, Q are points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

TP, TQ are tangents; the centroid of $\triangle TPQ$ lies on the ellipse; show that T lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=4.$$

98. P, Q are points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

TP, TQ are tangents; T lies on the line

$$\frac{x}{a}+\frac{y}{b}=1\ ;$$

show that the mid-point of the chord PQ lies on the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{x}{a}+\frac{y}{b}.$$

99. P, Q are variable points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

TP, TQ are tangents; the area of $\triangle TPQ$ is constant and equal to c; show that the locus of T has equation,

$$c^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)^3.$$

100. A, B, C are the points α, β, γ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

show that the tangents at A, B, C enclose an area

$$ab\tan\frac{\beta-\gamma}{2}\tan\frac{\gamma-\alpha}{2}\tan\frac{\alpha-\beta}{2}.$$

101. Tangents are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

show that the locus of their poles with respect to the circle

$$x^2+y^2=c$$

is the ellipse

$$a^2x^2+b^2y^2=c^2$$

102. Perpendiculars from the foci to a variable tangent of an ellipse meet the tangent at P, P'; prove that the locus of P and P' is the circle having as diameter the major axis of the ellipse.

103. TP, TQ are mutually perpendicular tangents to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

S, S_1 are the foci; find the equations of the parallels through the origin to the line pairs TP, TQ and SP, S_1P, and deduce that SP, S_1P are equally inclined to TP and to TQ.

104. Find the condition that the line

$$x\cos\alpha+y\sin\alpha=p$$

may be a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and find the co-ordinates of its point of contact.

Hence, or otherwise, show that the locus of the point of intersection of perpendicular tangents to an ellipse is a circle.

The tangents at two points P_1, P_2 of an ellipse are at right angles and intersect in Q. Prove that

$$p_1 \cdot QP_1=p_2 \cdot QP_2,$$

where p_1, p_2 are the lengths of the perpendiculars from the centre of the ellipse on the tangents at P_1, P_2 respectively.

(Cent. Welsh Bd. H.S.C.)

105. P is a variable point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

the normal at P meets the y-axis at N, and the parallel through N to the x-axis meets at Q the line joining P to the origin; show that Q lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{b^2}\right)^2.$$

106. The tangents at P, Q, two points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

intersect at the point (x, y); show that the normals at P, Q intersect at the point (ξ, η) where

$$\xi=\frac{a^2-b^2}{a^2}\cdot\frac{1-\dfrac{y^2}{b^2}}{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}x$$

and
$$\eta=-\frac{a^2-b^2}{a^2}\cdot\frac{1-\dfrac{x^2}{a^2}}{\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}}y.$$

107. If the normals at P, Q, points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

intersect on the line

$$a^2x+b^2y=0,$$

show that the tangents at P, Q intersect on the curve

$$x-y+xy\left(\frac{x}{a^2}-\frac{y}{b^2}\right)=0.$$

108. The normals at P, Q, points on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

intersect on the line

$$a^2x-b^2y=a^2-b^2;$$

show that the tangents at P, Q intersect on the curve

$$\frac{x^2}{a^2}(y+1)+\frac{y^2}{b^2}(x+1)=x+y.$$

109. Find the equation of the straight line joining the points on the ellipse $b^2x^2+a^2y^2=a^2b^2$, whose eccentric angles are α and β, and the co-ordinates of the point of intersection of the tangents at these points in terms of $\tan\frac{1}{2}\alpha$ and $\tan\frac{1}{2}\beta$.

The straight lines joining a point P on an ellipse to the foci of the ellipse meet the ellipse again at the points Q and R, and the tangents at Q and R meet at T Prove that T lies on the normal at P, and that PT is bisected by the minor axis. (Oxf. H.S.C.)

110. Show that the condition that the lines

$$l_1x+m_1y+n_1=0, \quad l_2x+m_2y+n_2=0$$

may be conjugate with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is that

$$a^2l_1l_2+b^2m_1m_2=n_1n_2.$$

111. The line joining $P(x_1, y_1)$ to the centre of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meets at Q the polar of P with respect to the ellipse ; show that, if P lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{k},$$

Q lies on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k.$$

112. The intercept made by the axes on the polar of P with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is equal to c ; show that P lies on the curve

$$\frac{a^4}{x^2}+\frac{b^4}{y^2}=c^2.$$

113. P is any point on the polar of the point (a, b) with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ ;$$

show that the inverse of P with respect to the circle

$$x^2+y^2=c^2$$

lies on the circle

$$x^2+y^2=c^2\left(\frac{x}{a}+\frac{y}{b}\right).$$

114. P is a variable point on the line

$$x=y\,;$$

the line through P parallel to the x-axis meets at Q the polar of P with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,;$$

show that the locus of Q has equation

$$b^2xy=a^2(b^2-y^2).$$

115. Tangents are drawn to the circle

$$x^2+y^2=c^2\,;$$

show that the locus of their poles with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}.$$

116. Tangents are drawn to the parabola

$$y^2=4cx\,;$$

show that their poles with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

lie on the parabola

$$a^2cy^2+b^4x=0.$$

117. P is any point on the line

$$y=b^2x\,;$$

the polar of P with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meets the major axis at Q; RP, RQ are parallel to the major and minor axes respectively; show that the locus of R has equation

$$xy=a^2b^2.$$

118. P is a variable point on the conic

$$xy = c^2;$$

the polar of P with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

meets the major and minor axes at Q and R; show that the locus of the mid-point of QR has equation

$$4c^2xy = a^2b^2.$$

119. P is a variable point on the director circle of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

the polar of P with respect to the ellipse cuts the latter at Q and R; find the equation of the locus of the mid-point of QR.

120. CP, CQ are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

show that the ordinate of P : the abscissa of $Q = b : a$, numerically.

121. CP, CQ are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

the chord PQ meets the co-ordinate axes at A, B and $CADB$ is a rectangle; show that D lies on the curve

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} = 2.$$

122. CP, CQ are conjugate semi-diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

the circles with CP, CQ as diameters intersect at R; show that R lies on the curve

$$a^2x^2 + b^2y^2 = 2(x^2 + y^2)^2.$$

123. Chords are drawn through a fixed point (x_1, y_1) on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

show that the mid-points of the chords lie on the ellipse

$$\frac{(2x-x_1)^2}{a^2}+\frac{(2y-y_1)^2}{b^2}=1.$$

124. Chords of an ellipse pass through a fixed point; show that their mid-points lie on an ellipse whose axes are in the same ratio as those of the given ellipse.

125. Show that the locus of the mid-points of normal chords of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

has equation

$$\left(\frac{x^2}{a^6}+\frac{y^2}{b^6}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2y^2}{a^2b^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)^2.$$

126. P, Q are points on the parabola

$$y^2=4ax,$$

whose polars with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

are parallel to conjugate diameters of this ellipse; show that the poles of PQ with respect to the parabola and the ellipse have loci respectively

$$b^2x+4a^3=0 \quad \text{and} \quad 4ax-b^2=0.$$

127. CP, CD are conjugate semi-diameters of an ellipse; the tangent at P meets any two parallel tangents at Q, R; prove that

$$QP\,.\,PR=CD^2.$$

128. A', B', C' are the mid-points of the sides BC, CA, AB of a triangle inscribed in an ellipse with centre O; P is any point on the ellipse, and the lines through P parallel to OA', OB', OC' meet BC, CA, AB at L, M, N respectively; show that L, M, N are collinear.

129. APB, CPD, two chords of an ellipse, are parallel to conjugate diameters; M is the mid-point of AC; show that MP and BD are parallel to conjugate diameters.

130. $\triangle ABC$ is inscribed in an ellipse; BT, CT are tangents to the ellipse; the line through T parallel to the tangent at A meets AB, AC at P, Q respectively; prove that $PT=TQ$.

131. $\triangle ABC$ is inscribed in an ellipse; AT, CT are tangents to the ellipse; the diameter which bisects AB meets BC at D; show that TD is parallel to AB.

132. PP', QQ' are mutually perpendicular normals to an ellipse at P and Q; show that the chords PQ, $P'Q'$ are parallel.

C. On the Hyperbola

133. NP is the ordinate of a variable point P on the hyperbola,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

the curve meets the x-axis at A, A'; show that

$$\frac{NP^2}{AN\,.\,A'N}$$

is constant.

134. P is the point (x, y) on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that the distances of P from the foci are $ex \pm a$.

135. B is a fixed point in the plane of a circle with centre A; P is a variable point on the circumference; BP meets the circle again at Q, and the parallel through B to AQ meets AP at R; show that the locus of R is an ellipse or a hyperbola according as B is inside or outside the circle.

136. P is the centre of a variable circle which touches each of two intersecting circles; show that the locus of P consists of an ellipse and a hyperbola.

137. Show that the point

$$\left\{\frac{a}{2}\left(t+\frac{1}{t}\right),\ \frac{b}{2}\left(t-\frac{1}{t}\right)\right\}$$

lies on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

and show that the equation of the chord joining the points t_1 and t_2 is

$$\frac{x}{a}(t_1t_2+1)-\frac{y}{b}(t_1t_2-1)=t_1+t_2.$$

138. P is a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

Q is the inverse of P with respect to the circle

$$x^2+y^2=c;$$

show that the locus of Q has equation

$$c^2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=(x^2+y^2)^2.$$

139. Prove that the tangents to the parabola $y^2=ax$, at the points (at_1^2, at_1), (at_2^2, at_2), intersect in the point $\{at_1t_2, \frac{1}{2}a(t_1+t_2)\}$.

The tangent to the parabola $y^2=ax$ at any point P, cuts the hyperbola $x^2-4y^2=a^2$ in the points U, V. Prove that if Q and R are the points of contact of the remaining tangents to the parabola from U and V, the chord QR touches a fixed circle whose centre is at the vertex of the parabola. (Oxf. H.S.C.)

140. Prove that the foot of the perpendicular from the focus of a hyperbola to either asymptote lies on the directrix.

141. P is a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

a line drawn in a fixed direction passes through P and meets the asymptotes at Q and R; show that $QP \, . \, PR$ is constant.

142. Show that the point $(a \tan \phi, b \sec \phi)$ lies on the hyperbola conjugate to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

143. P is a point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

Q is a point on the conjugate hyperbola, such that PQ subtends a right angle at the origin O; show that

$$\frac{1}{OP^2}-\frac{1}{OQ^2}=\frac{1}{a^2}-\frac{1}{b^2}.$$

144. The tangent at a point P on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

meets the conjugate hyperbola at A and B; show that $AP=PB$.

145. P is any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,;$$

tangents from P to the conjugate hyperbola touch the latter at Q and R; show that QR touches the given hyperbola.

146. The tangent at the point $(a, 0)$ to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

meets at Q the tangent at a variable point P; show that the locus of the mid-point of PQ has equation

$$4a^2xy^2=b^2(x-a)(x+2a)^2.$$

147. A variable tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

meets, at P and Q, the transverse axis and the tangent at the point $(a, 0)$; show that the locus of the mid-point of PQ has equation

$$x(4y^2+b^2)=ab^2.$$

148. P, Q are variable points θ, $\theta+2\alpha$ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that, if α is constant, the locus of the intersection of tangents at P and Q is the hyperbola

$$\frac{x^2}{a^2\cos^2\alpha}-\frac{y^2}{b^2}=1.$$

149. A, B are variable points θ, 2θ on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,;$$

show that the locus of the point of intersection of the tangents at A and B has equation

$$a^2y^2(3x+a)=b^2(2x+a)^2(x-a).$$

150. P is any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\,;$$

the tangent at P meets an asymptote at T, and the perpendiculars from T to the x- and y-axes meet these axes at Q and R; prove that P, Q, R are collinear.

151. The tangent at P, a variable point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

meets the asymptote

$$\frac{x}{a}-\frac{y}{b}=0$$

at Q; show that the locus of the mid-point of PQ has equation

$$4\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=3.$$

152. P, Q are any two points on a hyperbola; the tangents at P, Q meet one asymptote at R, S respectively and the other asymptote at T, V respectively; show that ST is parallel to RV.

153. A tangent to a hyperbola at a point P meets the asymptotes at Q, R; show that $QP=PR$.

154. A tangent to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

at the point P meets the asymptotes at Q, R; VQ, VR are parallel to the axes of the hyperbola; show that V lies on the rectangular hyperbola

$$xy=\pm ab.$$

155. Show that the polar of any point on the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

with respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

is a tangent to the hyperbola.

156. Show that the locus of the poles of tangents to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

with respect to the parabola

$$y^2=4cx$$

is the ellipse

$$\frac{x^2}{b^2}+\frac{y^2}{4c^2}=\frac{a^2}{b^2}.$$

157. A normal to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

meets the transverse and conjugate axes at P, Q respectively; show that the mid-point of PQ lies on the hyperbola

$$4(a^2x^2-b^2y^2)=(a^2+b^2)^2.$$

158. Find the co-ordinates of the point of intersection of the normals to the parabola $y^2=4ax$ at the points $(at_1^2, 2at_1)$, $(at_2^2. 2at_2)$.

If two normals to a parabola intersect on a fixed straight line, prove that the locus of the points of intersection of the corresponding tangents is an hyperbola, one of whose asymptotes is the tangent to the parabola at right angles to the fixed straight line.

(Oxf. H.S.C.)

159. P, Q are conjugate points with respect to the conics

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and } \frac{x^2}{a^2}-\frac{y^2}{b^2}=1;$$

show that P, Q lie on the x-axis and are inverse points with respect to the circle

$$x^2+y^2=a^2.$$

160. Show that the locus of the poles of tangents to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

with respect to the circle

$$x^2+y^2=c^2,$$

is the hyperbola

$$\frac{x^2}{b^2}-\frac{y^2}{a^2}=\frac{c^4}{a^2b^2}.$$

161. A, B are the fixed points $(-a, 0)$, $(a, 0)$; a variable circle passing through A and B meets the y-axis at C, D; AC, BD meet at P; show that the locus of P is the rectangular hyperbola

$$x^2-y^2=a^2.$$

162. P is a variable point on the hyperbola

$$xy=c^2;$$

Q is the inverse of P with respect to the circle

$$x^2+y^2=c^2:$$

show that the locus of Q has equation

$$(x^2+y^2)^2=c^2xy.$$

163. The lines

$$l_1x+m_1y+n_1=0, \quad l_2x+m_2y+n_2=0,$$

are conjugate with respect to the rectangular hyperbola

$$xy=c^2;$$

show that

$$2c^2(l_1m_2+l_2m_1)=n_1n_2.$$

164. Show that the feet of the four normals to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from the point (α, β) lie on a rectangular hyperbola which passes through the centre of the ellipse and the point (α, β) and whose asymptotes are parallel to the axes of the ellipse.

If the normals at the points A, B, D, E on the ellipse are concurrent, show that DE and the diameter conjugate to AB are equally inclined to the major axis of the ellipse. (Lond. H.S.C.)

165. Show that the feet of the four normals drawn from any point to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

lie on a rectangular hyperbola, and that, if the chord joining two of the points is

$$\frac{lx}{a}+\frac{my}{b}=1,$$

then the chord joining the other two is

$$\frac{x}{al}+\frac{y}{bm}+1=0. \qquad \text{(Oxf. and Camb. H.S.C.)}$$

166. P is a point on the parabola

$$ax^2=2c^2y;$$

show that the polar of P with respect to the parabola

$$y^2 = 4ax$$

touches the rectangular hyperbola

$$xy = -c^2.$$

167. P, Q are points on the hyperbola

$$xy = c^2,$$

such that their polars with respect to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are parallel to conjugate diameters of the latter; show that the pole of PQ with respect to the rectangular hyperbola lies on an asymptote of the other hyperbola.

168. P, Q are points on the rectangular hyperbola

$$xy = c^2,$$

such that their polars with respect to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are parallel to conjugate diameters of the latter; show that the locus of the pole of PQ with respect to the latter hyperbola is the asymptotes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

D. On Polar Equations

169. Given that the polar equation of an ellipse is

$$\frac{l}{r} = 1 - e \cos \theta;$$

show that the sum of the focal distances of any point on the ellipse is $2a$.

170. Given that the polar equation of a hyperbola is

$$\frac{l}{r} = 1 - e \cos \theta;$$

show that the difference of the focal distances of any point on the curve is $2a$.

171. Show that the chord of contact of tangents from the point (r_1, θ_1) to the circle, $r=a$, has polar equation

$$rr_1 \cos(\theta-\theta_1)=a^2.$$

172. The polar equation of a circle is

$$r^2-2\beta r\cos(\theta-\alpha)+\beta^2-a^2=0;$$

a variable line through the pole O meets the circle at A, B; show that $OA \,.\, OB$ is constant.

173. A variable line drawn from the pole to the circle

$$r^2-2\beta r\cos(\theta-\alpha)+\beta^2-a^2=0$$

is bisected at M; show that the locus of M is a circle, and find its equation and radius.

174. A variable line drawn from the pole O to the circle

$$r^2-2\beta r\cos\theta+\beta^2-a^2=0$$

meets the latter at A, B; P is the harmonic conjugate of O with respect to A and B; show that the locus of P is the straight line

$$\beta r\cos\theta=\beta^2-a^2.$$

175. Show that the chord joining the points on the circle

$$r=2a\cos\theta$$

with vectorial angles α and β has equation

$$r\cos(\alpha+\beta-\theta)=2a\cos\alpha\cos\beta.$$

176. Show that the normal to the circle

$$r=2a\cos\theta,$$

at the point with vectorial angle α has equation

$$r\sin(2\alpha-\theta)=a\sin 2\alpha.$$

177. Show that the equation of the pair of tangents from the point (r_1, θ_1) to the circle $r=2a\cos\theta$ is

$$(r_1-2a\cos\theta_1)(r-2a\cos\theta)$$
$$=rr_1\left\{\cos(\theta-\theta_1)-a\left(\frac{\cos\theta}{r_1}+\frac{\cos\theta_1}{r}\right)\right\}^2.$$

178. Show that the polar of the point (r_1, θ_1) with respect to the circle

$$r=2a\cos\theta$$

has equation

$$a\left(\frac{\cos\theta}{r_1}+\frac{\cos\theta_1}{r}\right)=\cos(\theta-\theta_1).$$

179. Show that the tangent at the point with vectorial angle α on the parabola

$$r \sin^2\theta = 4a \cos \theta$$

has equation

$$\frac{4a}{r} = 2 \tan \alpha \sin \theta - \tan^2\alpha \cos \theta.$$

180. If r_1, r_2 are mutually perpendicular radii vectores of points on the parabola

$$\frac{l}{r} = 1 - \cos \theta,$$

show that

$$l^2\left(\frac{1}{r_1{}^2} + \frac{1}{r_2{}^2}\right) - 2l\left(\frac{1}{r_1} + \frac{1}{r_2}\right) + 1 = 0.$$

181. Show that the part of any tangent to a conic intercepted between the directrix and the point of contact subtends a right angle at the focus.

182. S is the focus of the central conic

$$\frac{l}{r} = 1 - e \cos \theta;$$

the axis through S meets the conic at A and the directrix at Z; show that AZ has length

$$\pm\frac{l}{e(1 \pm e)}.$$

183. Show that the equations

$$\frac{e}{r} = -\frac{\cos \theta}{a} \pm \frac{\sin \theta}{b}$$

represent the asymptotes of the hyperbola

$$\frac{l}{r} = 1 - e \cos \theta,$$

if $2a$, $2b$ are the lengths of the transverse and conjugate axes.

184. Show that the tangents at the points on the parabola

$$\frac{l}{r} = 1 - \cos \theta,$$

whose vectorial angles are α, β, γ, intersect each other at points on the circle

$$r = -\frac{l}{2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}} \sin\left(\theta - \frac{\alpha + \beta + \gamma}{2}\right).$$

185. A circle passing through the focus of the conic

$$\frac{l}{r}=1-e\cos\theta$$

cuts the conic in points whose radii vectores are r_1, r_2, r_3, r_4; show that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}=\frac{2}{l}.$$

186. The perpendicular from the focus S to the directrix of a conic meets the conic at A; show that the foot of the perpendicular from S to any tangent lies on the tangent at A, if and only if the conic is a parabola.

187. Four normals are drawn from a point P to the conic

$$\frac{l}{r}=1-e\cos\theta;$$

the vectorial angles of P and of the feet of the normals are ϕ, α, β, γ, δ respectively; show that

$$\alpha+\beta+\gamma+\delta-2\phi=(2n+1)\pi,$$

where n is integral.

E. On the General Equation

Determine the nature of the following loci:

188. $3x^2+2xy+3y^2+8x+4=0.$

189. $3x^2+3y^2+2x-3y-1=0.$

190. $4x^2+4y^2-8x-12y+13=0.$

191. $2x^2-12xy-7y^2-10y-3=0.$

192. $3x^2+8xy-3y^2-x+7y-2=0.$

193. $3x^2+8xy-3y^2+4x+2y=0.$

194. $x^2-4xy+4y^2-2y+2=0.$

195. $2x^2+8xy+8y^2-x-2y-3=0.$

196. Find the centre and the equations and lengths of the axes of the ellipse

$$8x^2-12xy+17y^2+8x-56y-48=0.$$

197. Find the centre, the equations and lengths of the axes and the equations of the asymptotes of the hyperbola

$$x^2+6xy-7y^2-8x+8y-20=0.$$

198. Find the centre, the equations and length of the axes and the equations of the asymptotes of the rectangular hyperbola

$$2x^2+3xy-2y^2-4x-3y-23=0.$$

199. Find the focus and directrix of the parabola

$$x^2-4xy+4y^2-24x-2y+69=0.$$

200. The coordinates of any point of a parabola referred to oblique axes crossing at an angle ω are

$$x=at^2+2bt+c, \quad y=pt^2+2qt+r,$$

where t is a parameter; find the condition that the tangents at the points given by the values t_1, t_2 of t should be perpendicular.

Hence, or otherwise, show that when $\omega=60°$ the line given by

$$lx+my+n=0$$

passes through the focus of the parabola given by

$$x=-t^2+2t+2, \quad y=2t^2-1,$$

if $5l-m+3n=0$. (Camb. H.S.C.)

201. Find the equation of the rectangular hyperbola whose vertices are at the points (2, 3), (4, 7). (Camb. H.S.C.)

ANSWERS TO EXERCISES

PART I

Page 8.

3. $(-1, 1)$. **4.** $(1, -2)$.

8. (i) -2; (ii) -5; (iii) 3; (iv) -4; (v) 7; (vi) 4.

10. $-\frac{1}{2}, 2$. **11.** 8. **13.** $(0, -1), (0, 11), 12$.

14. $(1, \sqrt{3}), (-2, 2), (0, -3), (\frac{1}{2}\sqrt{3}, -\frac{1}{2})$.

15. $\left(2, \frac{\pi}{6}\right), \left(\sqrt{2}, \frac{7\pi}{4}\right), \left(2, \frac{4\pi}{3}\right), \left(1, \frac{\pi}{2}\right)$. **17.** $5, 13, 10, a$.

18. $150°$. **20.** $3\sqrt{5}$. **22.** $2{\cdot}9$ approx.

23. (i) $3, 3$; (ii) $5, 0$; (iii) $-6, -5$; (iv) $7, -3$. **24.** 3.

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1. (i) 5; (ii) 13; (iii) $3\sqrt{10}$; (iv) $5\sqrt{5}$. **2.** $2\sqrt{2}$. **3.** $4\sqrt{2}, 12\sqrt{2}$.

7. $(3, 4), (-1, 4), (-2, 2)$. **10.** $2\sqrt{10}, 2\sqrt{26}$. **11.** $(3, 2)$.

12. $(-3, 1), (5, -11)$. **13.** $(12, 14), (-12, -10)$.

14. $\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}$.

15. (i) $\frac{3}{4}$; (ii) $\frac{1}{3}$; (iii) -1; (iv) $-\frac{1}{pq}$; (v) $\frac{2}{p+q}$; (vi) $-\frac{b}{a}\cot\frac{\theta+\varphi}{2}$.

22. $3:1$. **24.** $\frac{\pi}{4}$. **25.** $\pm\frac{7}{6}$.

26. $-\frac{7}{4}, 2, \frac{3}{2}$. **27.** $\frac{4\sqrt{3}}{15}$.

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1. (i) 6; (v) 1; (viii) $\frac{ab}{2}\sin(\beta-\alpha)$;

(ii) 31; (vi) $\frac{1}{2}(a^2-b^2)$; (ix) $\frac{1}{2}\sin\alpha$;

(iii) 11; (vii) $\frac{a-b}{2c}$; (x) $\frac{1}{2}ab\sin\theta$.

(iv) 4;

3. 8. **4.** $z-3$. **5.** $+1$ or -3. **6.** $\frac{1}{2}b$.

7. (i) 12, (ii) $\frac{5}{2}$; (iii) $\frac{11}{2}$; (iv) $\frac{27}{2}$; (v) $-\frac{35}{2}$; (vi) 32.

8. $6\frac{1}{4}$. **10.** (i) 17; (ii) $\frac{37}{2}$; (iii) $\frac{69}{2}$; (iv) 5.

12. $\frac{1}{2}(2a-1)$. **13.** $\frac{1}{2}(2p^2-3p-2)$. **14.** 5.

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1. (i) $x = \pm y$; (ii) $x+y=6$; (iii) $5x-4y=0$; (iv) $x^2+y^2=25$; (v) $4x^2-y^2=0$.

5. $x+y=7$, $x+2y=11$. **14.** $(0, -2)$, $(0, -\frac{1}{2})$.

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1. (i) $3x-4y+2=0$; (ii) $5x+8y+16=0$; (iii) $x+y+3=0$; (iv) $5x-3y-15=0$; (v) $6x+4y+3=0$; (vi) $x\cos\alpha+y\sin\alpha=p$; (vii) $y=mx+c$.

2. (i) on x-axis, $-2, 6$; on y-axis, $3, -9$;
(ii) „ „ $-2, -4$; „ „ $-5, -10$;
(iii) „ „ $-\frac{c}{a}, 1$; „ „ $-\frac{c}{b}, \frac{a}{b}$;
(iv) „ „ $p\sec\alpha, a\sec\alpha$; „ „ $p\operatorname{cosec}\alpha, a\operatorname{cosec}\alpha$.

3. $y=mx+\dfrac{a}{m}$. **4.** $x\cos\theta+y\sin\theta=a$. **6.** $4x+y-7=0$.

7. $x-2y-8=0$. **8.** $2x-y-14=0$. **9.** $3x-y-9=0$.

11. $a=3$ or $-\frac{25}{3}$. **13.** $-\frac{4}{3}$.

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1. (i) $5x-6y+9=0$; (ii) $4x+5y-12=0$; (iii) $\sqrt{3}x-y=4(2+\sqrt{3})$; (iv) $x+y-3=0$; (v) $3x-4y+18=0$; (vi) $\dfrac{x}{a}+\dfrac{y}{b}=1$; (vii) $y=2mx+\dfrac{a}{2m}$; (viii) $y=mx+a\sqrt{1+m^2}$.

2. (i) $4x-3y+12=0$; (ii) $2x-y-1=0$; (iii) $3x-2y-16=0$; (iv) $8x+6y-13=0$; (v) $x+y=a+b$; (vi) $y=mx+c$; (vii) $3y=4mx+\dfrac{2a}{m}$; (viii) $x+2t^2y=3ct$.

3. $x+2y-12=0$.

4. $2x-5y+24=0$, $7x+y+10=0$, $9x-4y-3=0$.

5. $9x+8y+12=0$. **6.** $3x-y-3=0$.

7. $x-2y+5=0$, $2x+y-10=0$. **8.** $x+y-6=0$.

9. $y=mx+\dfrac{a}{m}$, $y=-mx-\dfrac{a}{m}$.

10. Points are (1, 1), (−1, −1), (0, 0).

Equations are $\dfrac{x-1}{+\frac{3}{5}}=\dfrac{y-1}{\frac{4}{5}}=r,\ \dfrac{x+1}{\frac{12}{13}}=\dfrac{y+1}{-\frac{5}{13}}=r,\ \dfrac{x}{\frac{\sqrt{3}}{2}}=\dfrac{y}{\frac{1}{2}}=r.$

11. (9, −1) or (−3, −6).

12. 13.

13. (4, 5) and (−8, −11).

14. $5x+y-7=0$.

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3. $x+y=0$.

4. (i) parallel ; (ii) $\tan\theta=\pm 1$; (iii) parallel ; (iv) perpendicular ; (v) $\tan\theta=\pm\frac{7}{4}$; (vi) perpendicular ; (vii) $\tan\theta=\pm\frac{14}{5}$; (viii) $\tan\theta=\pm\frac{7}{26}$.

5. (i) $3x+4y-4=0$; (ii) $6x+5y+65=0$.

6. $3x-2y+18=0$; (−4, 3).

7. $x\cos(\alpha-\theta)+y\sin(\alpha-\theta)=a$.

8. $a\cos\theta\,.\,x+b\cot\theta\,.\,y=a^2+b^2$.

9. $2x-3y-6=0,\ x+3y-12=0$; (6, 2).

10. 5.

11. $\left(\dfrac{a+b}{2},\ \dfrac{b-a}{2}\right)$.

12. $3x=4y$.

13. $(3\frac{1}{4},\ 1\frac{1}{2})$.

14. 5.

15. (0, −1).

17. (3, 2).

18. $\frac{81}{2}$.

19. $(-\frac{5}{3},\ -\frac{8}{3})$.

21. $y=mx+\dfrac{a}{m},\ y=-\dfrac{x}{m}-am$.

26. (1, 0).

29. 6.

30. $\frac{7}{4}$.

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1. (i) $x\cos\alpha+y\sin\alpha=1$ where $\tan\alpha=\frac{4}{3}$ and α is acute ;
(ii) $x\cos\alpha+y\sin\alpha=2$, where $\alpha=120°$;
(iii) $x\cos\alpha+y\sin\alpha=2\sqrt{2}$, where $\alpha=45°$;
(iv) $x\cos\alpha+y\sin\alpha=2$, where $\tan\alpha=\frac{15}{8}$ and α is 3rd quadrant angle ;
(v) $x\cos\alpha+y\sin\alpha=2$, where $\tan\alpha=-\frac{12}{5}$ and α is obtuse ;
vi) $x\cos\alpha+y\sin\alpha=1$, where $\tan\alpha=-\frac{9}{40}$ and α is 4th quadrant angle.

2. 10.

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9. $\dfrac{(k+1)x}{a}+\dfrac{(l+1)}{b}y=1$.

12. externally as 5 : 2 ; internally as 1 : 2.

14. 3 : 1.

15. −5 : 4.

17. internally in ratio 14 : 11.

20. $(\frac{3}{5},\ \frac{11}{5}),\ (\frac{4}{5},\ \frac{18}{5}),\ (\frac{1}{5},\ -\frac{3}{5})$.

21. 14.

24. (14, 4).

27. $x(1+c)-3y+6=0,\ x(1+c)-5y+20=0$; $(\frac{15}{2}, 7)$.

30. $x+3y-1=0$; 10. **31.** $2\sqrt{10}$.

32. $4x-3y-9=0$ and $3x-4y-5=0$.

33. $\frac{5}{2}\sqrt{5}$; $11x+2y-37=0$. **37.** $4x+y+3=0$. **40.** $2x-y-1=0$.

41. $\dfrac{3t}{2-3t}$. **43.** $x-4y+3=0$. **44.** $-\dfrac{\sqrt{3}}{2}$; 6, 4.

45. $13x-3y-7=0$. **46.** $\frac{5}{12}$.

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1. (i) 3 ; (ii) 2 ; (iii) $2\sqrt{5}$; (iv) a ; (v) a ; (vi) $\dfrac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$.

2. $\frac{1}{2}$. **5.** 10. **7.** 2.

12. (i) $x=0,\ y=1$;
(ii) $7x-7y+5=0,\ x+y+1=0$;
(iii) $24x+8y+7=0,\ 16x-48y+23=0$.

17. $12x-5y-109=0$, $(7, -5)$, 13. **18.** $3x+4y-21=0$.

20. $4x+5y+1=0,\ y+1=0$.

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1. $13x-22y=0$. **2.** $4x-3y+1=0$. **4.** $y+2=0$.

5. $x+y=0,\ x-y+2=0$. **6.** $(-5, 0)$.

7. $(3, 4)$. **8.** $(-1, 2)$.

10. line parallel to x-axis, concurrent with $ax+by+c=0$, $y=x+c$.

11. $x+3y-7=0$. **12.** $3x-y-2=0,\ 2x+y-3=0$; $45°$.

16. $x+2y+3=0,\ 2x-y-4=0,\ 7x+9y+11=0$.

20. 3. **21.** 4. **22.** $6a-4b=1$.

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1. (i) $3x^2-5xy+2y^2=0$;
(ii) $2x^2-5xy+3y^2+5x-7y+2=0$;
(iii) $6x^2+5xy-4y^2-10x+5y=0$;
(iv) $2xy-y^2-2x+2y-1=0$.

2. (i) $2x-y=0,\ x+y=0$;
(ii) $x-2y+1=0,\ 3x+y-2=0$;
(iii) $x+3y-1=0,\ y=4$;
(iv) $x+y+1=0,\ x-y+1=0$;
(v) $3x-2y+1=0,\ 2x+y-2=0$.

3. (i) $2x^2-3xy-2y^2=0$; (iii) $6x^2+xy-y^2=0$;
(ii) $6x^2-5xy-6y^2=0$; (iv) $xy=0$.

4. $(3, 1)$. **5.** $5x-7y=0$. **9.** $9\frac{1}{2}$. **11.** 10.

12. (i) $k=12$; $4x-3y+1=0$, $3x+2y-1=0$.
(ii) $k=-5$; $3x+y-4=0$, $2x-5y+3=0$.
(iii) $k=-3$; $4x+y-3=0$, $3x+4y+1=0$.
(iv) $k=16$ or 32; $x+y+2=0$, $x+5y+6=0$.
or $x+y+6=0$, $x+5y+2=0$.
(v) $k=10$ or 14; $x+y+2=0$, $3x+y+4=0$
or $x+y+4=0$, $3x+y+2=0$.
(vi) $k=-1$ or 4; $x-2y+1=0$, $2x+3y+1=0$
or $2x-2y+1=0$, $x+3y+1=0$.

13. (i) $\frac{1}{7}$; (ii) 5; (iii) $\frac{11}{7}$.

16. (i) $x^2-xy-6y^2=0$; (ii) $x^2-3xy+2y^2=0$; (iii) $3x^2-xy-2y^2=0$.

20. (i) $11x^2-6xy-11y^2=0$; (ii) $x^2-9xy-y^2=0$; (iii) $11x^2-26xy-11y^2=0$.

25. 5. **26.** $x^2+3xy+2y^2=0$.

29. (i) $(1, 2)$; (ii) $(-2, 3)$; (iii) $(-1, -3)$; (iv) $(\frac{3}{2}, -\frac{1}{2})$.

31. $\frac{15}{29}$.

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1. $(3, 1)$, $(-1, 2)$, $(2, 0)$. **2.** $(-2, -1)$. **3.** $2, -1$. **4.** $8, 17$.

5. (i) $x-3y+3=0$; (ii) $2x+3y=0$; (iii) $2x-3y+3=0$.

6. $x-2y+5=0$.

7. (i) $3x^2-5xy+y^2+7x-8y+3=0$; (ii) $4x^2-y^2+8x-2y+3=0$; (iii) $x^2-3xy+2y^2=0$.

8. $3x^2+xy-2y^2+(6\alpha+\beta-8)x+(\alpha-4\beta+7)y$
$+3\alpha^2+\alpha\beta-2\beta^2-8\alpha+7\beta-3=0$, $(1, 2)$.

9. $4x+3y+25=0$, 5. **10.** $x-2y+2\sqrt{2}=0$. **11.** $3x^2-13xy+12y^2=0$.

12. $xy+3y^2=0$. **13.** $x=p$. **14.** $5xy+8x=0$. **15.** $x^2-y^2=0$.

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1. $-a(\cos\theta+\sin\theta)$, $b(\cos\theta-\sin\theta)$. **7.** $(-\frac{3}{2}, 4)$.

8. $(-6, -1)$, $(3, 2)$, $4\sqrt{10}$, $4\sqrt{5}$, $3\sqrt{10}$, $6\sqrt{5}$.

9. $(-2, 1)$, $(6, 5)$, $(0, -4)$. **10.** $(7, 5)$, $(11, 0)$. **11.** $1:2$.

13. $\frac{1}{4}(a^2-b^2)$. **15.** $1:4$. **18.** $\frac{1}{4}(9-2\sqrt{3})$, $\frac{1}{2}(8+\sqrt{3})$.

19. $(-1, 0)$. **20.** $(-9, 6)$. **21.** $7x-7y+4=0$.

23. $\left(\frac{5k-2}{2}, \frac{5-2k}{2}\right)$. **24.** $13x+y-65=0$, $13x+y-25=0$.

25. $x+3y-20=0$. **28.** 1. **30.** $x+y+3=0$, $x-y=0$.

36. $(-2, 1)$. **37.** $(1, -2)$, 5. **38.** $(-4\frac{1}{2}, -\frac{1}{4})$, $(2, 3)$.

40. $x-2y+2a=0$, $x-2y+4a=0$. **41.** $(0, 3)$, $(1, 1)$. **44.** $(2, 8)$, $(10, 0)$.

47. $3x-4y-28=0$, $3x+y+7=0$, $x+2y-6=0$, $(0, -7)$, $(8, -1)$.

48. $2x-y+4=0$, $4x+3y-2=0$, $\sqrt{5}$. **49.** $3\frac{1}{2}$.

50. $3x-y+3=0$, $x-y-1=0$. **52.** $x-2y-3=0$. **54.** 2 or 22.

59. $(-\frac{8}{5}, \frac{4}{5})$, $(-4, 4)$, $(-1, 0)$. **60.** $(0, 1)$, $(-4, 2)$.

61. $(\frac{2}{5}, 4)$. **62.** $(1, 1)$. **63.** 54. **64.** $\dfrac{l-1}{m-1}, -\dfrac{l+1}{m+1}$.

67. $$x \cos\frac{\alpha+\alpha_1}{2}+y\sin\frac{\alpha+\alpha_1}{2}=\frac{1}{2}(p+p_1)\sec\frac{\alpha-\alpha_1}{2},$$

$$x\cos\frac{\alpha+\alpha_1+\pi}{2}+y\sin\frac{\alpha+\alpha_1+\pi}{2}=\frac{1}{2}(p-p_1)\operatorname{cosec}\frac{\alpha-\alpha_1}{2}.$$

69. $2x-y-1=0$, $x+2y-13=0$.

70. $(\frac{6}{5}, -\frac{1}{10})$, $(-\frac{2}{5}, -\frac{13}{10})$, $(0, \frac{3}{2})$, $(-\frac{8}{5}, \frac{3}{10})$. **71.** $x+7y+1=0$.

72. $x=0$. **73.** $(1, 1)$. **74.** $(-1, 2)$. **76.** -3.

81. $(0, 2)$. **82.** $\frac{9}{2}$. **83.** $a=2$, $c=-3$. **84.** $-\frac{17}{2}$, 3, 45°.

89. $2:1$, $1:2$. **95.** (α, β). **101.** $(2, 1)$, $(1, -7)$, $(-2, 4)$.

102. $x^2-y^2=2c^2$. **103.** $x^2-y^2=8$.

106. $x^2+y^2=8$. **107.** $y^2=x$.

PART II

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1. (i) $x^2+y^2=4$, (iv) $x^2+y^2=5$,
(ii) $x^2+y^2-4x+6y+4=0$, (v) $x^2+y^2+2x-4y=0$,
(iii) $4x^2+4y^2+8x-16y-25=0$, (vi) $x^2+y^2-6x-2y+5=0$.

2. (i) $(0, 0), 2\sqrt{2}$; (iii) $(1, -2), \sqrt{5}$; (v) $(\frac{3}{4}, -\frac{5}{4}), \frac{3}{2}$;
(ii) $(0, 0), \frac{1}{2}\sqrt{6}$; (iv) $(-1, 2), 2$; (vi) $(1, 3), \sqrt{5}$.

3. $x^2+y^2=221$. **4.** $-1, -3, 5$.

5. $x^2+y^2+6x-2y-15=0, (-3, 1), 5$. **6.** $x^2+y^2-2y-4=0, (0, 1)$.

7. $(2, -1), (4, 0)$. **9.** $(3, 4), (-4, 3)$. **10.** $(3, 1), (-2, 6)$.

11. $\sqrt{10}$. **12.** 5. **15.** $x^2+y^2=65, 4\sqrt{5}$. **16.** $(-5, -7)$.

19. $1:1, 1:3$. **21.** $x^2+y^2+30x-36y+29=0$.

22. $(-8, -1), (-\frac{4}{5}, \frac{7}{5})$. **23.** $x^2+y^2-8y-9=0$.

24. $x^2+y^2-5x-2y+1=0$. **25.** $x^2+y^2-4x-4y-2=0$.

26. $x^2+y^2+10y=0, x^2+y^2-2x-4y-20=0$. **27.** $(4, 2), (-5, -1)$.

28. $(1, 2), (-3, -2)$. **29.** $(-1, -2), (3, 1)$.

38. $x^2+y^2+2fy=0$; circle passing through $(0, a), (0, -a)$.

39. $x^2+y^2-ax-by=0$. **40.** $b(x^2+y^2)-(a^2+b^2)y=0$.

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1. (i) $3x-4y+25=0$, (iv) $2x+3y=0$, (vii) $x-4y-9=0$.
(ii) $6x-2y-5=0$, (v) $2x+3y-22=0$,
(iii) $8x+6y+25=0$, (vi) $2x-y-9=0$,

2. (i) $4x+3y=0$, (iv) $3x-2y=0$, (vii) $4x+y-2=0$.
(ii) $x+3y=0$, (v) $3x-2y-7=0$,
(iii) $3x-4y=0$, (vi) $x+2y+3=0$,

3. (i) $(3, -4)$; (ii) $(0, 0)$; (iii) $(4, 2)$; (iv) $(4, 4)$; (v) $(-1, 2)$.

4. $x^2+y^2-2x-4y+1=0$. **5.** 4. **11.** $(a, 0)$. **15.** $4, (2, 0)$.

16. (i) ± 5; (iii) ± 1; (v) $3, \frac{1}{9}$; (vii) $2, -38$;
(ii) ± 3; (iv) 2; (vi) 2, 12; (viii) $-a, 9a$.

20. $x+y-8=0$, $x+y+4=0$. **21.** 8, 19. **24.** $4x-3y\pm25=0$.

25. $2x+y+4=0$, $2x+y-6=0$. **26.** $4x-3y+16=0$.

27. $x+2y+6=0$, $x+2y-4=0$. **28.** $x^2+y^2=13$.

29. $x^2+y^2-8x\pm10y+16=0$.

30. $x^2+y^2+4x-21=0$, $x^2+y^2-16x+39=0$.

32. (i) $x-5y+13=0$, $(2, 3)$, $(-3, 2)$;
(ii) $x-5y-4=0$, $(4, 0)$, $(-1, -1)$;
(iii) $x+3y+12=0$, $(-3, -3)$, $(0, -4)$;
(iv) $2x-y+2=0$, $(-\frac{3}{2}, -1)$, $(-\frac{1}{2}, 1)$.

34. $x+7y=25$, $3x-4y+25=0$, $4x+3y-25=0$.

35. $x+y-4=0$, $2x-y+4=0$, $x-2y-1=0$.

36. (i) $5y^2-12xy=0$, (iii) $x^2\cos^4\theta-2xy\sin\theta\cos\theta+y^2\sin^4\theta=0$.
(ii) $3x^2+4xy=0$,

37. (i) $2x^2+5xy+2y^2-5x+5y-25=0$,
(ii) $3x^2-10xy+3y^2+28x-4y-20=0$,
(iii) $2x^2-5xy+2y^2-2x+7y-4=0$.

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1. (i) $4x-4y+1=0$, (iii) $6x-4y-9=0$, (v) $y=mx+c$.
(ii) $6x+2y-9=0$, (iv) $2gx+2fy+g^2+f^2+c=0$.

2. (i) $(4, 1)$, (ii) $(-2, -5)$, (iii) $(0, -3)$, (iv) $(0, 3)$, (v) $(0, 0)$.

9. $-4, 3$. **13.** $x+y-2=0$. **16.** $(-\frac{1}{3}, \frac{2}{3})$. **18.** $x+2y-10=0$.

19. $(1, 2)$. **20.** $(-12, 1)$. **21.** $(-2, -1)$, $(-1, -2)$.

30. $(-2, -4)$, $4x+3y+20=0$. **35.** $2x+2y+1=0$.

38. $x^2+y^2+9x-3y=0$. **39.** $x^2+y^2-2x-6y+9=0$.

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4. $-4, 5\frac{1}{2}$. **5.** Outside, inside, on. **6.** 5. **8.** $(-\frac{5}{2}, \frac{5}{2})$, $\frac{1}{2}\sqrt{74}$.

9. $(1, -3)$ or $(-\frac{17}{5}, \frac{29}{5})$.

10. (i) $x-3y+8=0$, (ii) $x-3y+6=0$, (iii) $12x+9y-1=0$. **11.** $4\sqrt{2}$.

12. $(1, 0)$. **15.** $x^2+y^2-2x-4y-20=0$. **17.** $2x-y-1=0$.

18. $(-1, 0)$, $(1, 0)$, $(-2, 0)$, $(2, 0)$, $(-3, 0)$, $(3, 0)$; 1, 1, 2, 2, 3, 3.

19. $(0, 0)$, $(-1, 0)$, $(1, 0)$, $(-2, 0)$, $(2, 0)$; 2, $\sqrt{5}$, $\sqrt{5}$, $2\sqrt{2}$, $2\sqrt{2}$.

20. $(-2, 0)$, $(2, 0)$, $(-\frac{5}{2}, 0)$, $(\frac{5}{2}, 0)$, $(-3, 0)$, $(3, 0)$, $(-4, 0)$, $(4, 0)$; 0, 0, $\frac{3}{2}$, $\frac{3}{2}$, $\sqrt{5}$, $\sqrt{5}$, $2\sqrt{3}$, $2\sqrt{3}$.

21. $x^2+y^2+x-3y-4=0$. **22.** $x^2+y^2+x+y-2=0$.

23. $(-\frac{1}{2}, \frac{1}{2})$, $x^2+y^2-5x+2y-4=0$.

24. $x^2+y^2-5x-4y+4=0$, $x^2+y^2+75x+36y-76=0$.

25. $x^2+y^2-2y-8=0$, $x^2+y^2+3x-5y+4=0$.

29. $x^2+y^2+2x+2y-23=0$, $x^2+y^2-9x+14=0$.

30. $x^2+y^2+2x-2y-8=0$, $x^2+y^2+x-9y+18=0$.

33. $x^2+y^2-3x-5y+1=0$. **34.** $4x^2+4y^2+2y-29=0$.

35. $x^2+y^2-4x+6y+4=0$. **36.** $x^2+y^2+2x-8y+5=0$.

38. $3x^2+3y^2+4x+2y-15=0$.

39. $x^2+y^2-6y-1=0$, $9x^2+9y^2+26y-9=0$. **40.** $x^2+y^2-6y-4=0$.

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1. (i) $(1, -2)$, 2; (ii) $(\frac{3}{2}, -1)$, $\frac{1}{2}\sqrt{13}$; (iii) $(\frac{3}{2}, \frac{5}{2})$, 3;
(iv) $\left(a \cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2},\ a\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}\right)$, $a\sin\frac{\theta-\varphi}{2}$.

5. $\sqrt{2}$. **6.** $3\sqrt{5}$. **7.** $(2, 3)$. **8.** 5. **10.** 3, 4. **11.** $x^2+y^2+4x-4y+3=0$.

14. $x^2+y^2-ax-by=0$; a, b; $x^2+y^2-ax=0$, $x^2+y^2-by=0$.

15. (i) circle, centre $(a, 0)$, radius a; (ii) circle, centre (a, α), radius a.

17. Chord is trisected. **19.** $4x+3y-24=0$, $4x-3y=0$. **26.** 2 or $-\frac{1}{6}$.

32. $x^2-y^2=0$. **34.** $x-2y+8=0$, $x-2y-2=0$. **35.** $4\sqrt{5}$.

36. $4x-3y+6=0$, $4x-3y-44=0$. **37.** $x^2+y^2+2x+4y-8=0$, $2\sqrt{13}$.

38. $x-2y+5=0$, $x-2y-5=0$, $3x-y=0$, $x+3y=0$. **39.** $2x+y-4=0$.

40. $(-2, -1)$, $(-4, -1)$, $(-1, -4)$; $(-3, -3)$.

46. $x^2+y^2+4x-4y-17=0$. **47.** $2x^2+2y^2-5x-10y+15=0$.

48. $x^2+y^2+4x-2y-5=0$, $x^2+y^2-x-7y+10=0$.

49. $x^2+y^2-10x-4y+4=0$, $3x-4y+18=0$, $(0, 4\frac{1}{2})$.

53. $x^2+y^2-4x-6y+9=0$, $9(x^2+y^2)=25(4x+6y-25)$.

54. $x^2+y^2+4x-8y-5=0$, $4x^2+4y^2+16x-7y-20=0$.

55. $x^2+y^2-6x-14y+33=0$, $x^2+y^2-6x+6y-7=0$.

56. $x^2+y^2+4x-6y+8=0$, $5x^2+5y^2-52x-6y-152=0$.

57. $2x-y-4=0$. **58.** $2x+y+4=0$, $x-2y+7=0$.

59. $x^2+16xy-11y^2-40x-20y+100=0$.

60. $2x-y-1=0$, $x-2y+1=0$, $2x+y+5=0$, $x+2y+5=0$.

62. (i) $4xy+3y^2=0$, (ii) $3x^2-8xy-3y^2=0$, (iii) $13x^2-20xy-8y^2=0$.

63. (i) $2x+y-5=0,\ x-2y-5=0,$ (iv) $2x+y+2=0,\ x-2y-9=0,$
(ii) $3x+2y-13=0,\ 2x-3y-13=0,$ (v) $y=0,\ 4x+3y-12=0,$
(iii) $x+3=0,\ y+6=0.$

64. $2x+6y+17=0.$ **65.** $2x+y=0.$

68. $x+5y+1=0,\ 7x-y-5=0,\ x+y-3=0.$ **71.** $x+5y-13=0.$

72. $x^2+y^2+3x+3y=0.$ **74.** $(\frac{1}{2},\ -1).$ **75.** $(2,\ \frac{3}{2}).$ **78.** $x-y+8=0.$

80. $4x^2+4y^2+2x-3y=0.$ **82.** $(1, 2),\ (\frac{57}{29},\ -\frac{12}{29}).$ **83.** $(2, 3).$

85. $x^2+y^2-4y-1=0.$ **86.** $x^2+y^2-3x-y=0.$ **87.** $3x+4y+20=0.$

89. $x^2+y^2-7x-4y+10=0.$ **90.** $x^2+y^2-3x-5y+6=0.$

91. $(3, 3).$ **92.** $x^2+y^2-2x-6y-8=0.$

93. $x^2+y^2+gx+fy=0.$ **94.** $12\frac{1}{2},\ 2x^2+2y^2-12x-8y+1=0.$

95. $3x^2+3y^2+6x-18y-13=0.$ **96.** $x^2+y^2-8x-10y+31=0.$

98. $2xy-y^2-4x-y+6=0,\ 3x^2+4xy+y^2-20x-8y+12,$ the first pair intersecting inside, the second outside the circle.

101. $7x^2+7y^2+31x-5y+12=0.$

PART III

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1. $x^2-2xy+y^2+6x-2y+3=0$.
2. $x^2-4xy+4y^2-24x-12y+24=0$.
3. $4x^2+4xy+y^2-4x+8y-4=0$.
4. $x^2+2xy+y^2-4x+4y+4=0$.
5. $y^2-2x+1=0$.
6. $7x^2-2xy+7y^2-18x-34y+39=0$.
7. $14x^2-4xy+11y^2+2x+34y+14=0$.
8. $14x^2-8xy+14y^2-4x-4y-1=0$.
9. $9x^2-4xy+6y^2-20x-20y+20=0$.
10. $3x^2+4y^2+4x=0$.
11. $x^2-4xy+y^2+2x-2y=0$.
12. $5x^2-24xy+12y^2+14x+8y-19=0$.
13. $x^2+18xy+y^2-18x-18y+9=0$.
14. $11x^2+64xy+59y^2+10x+10y-10=0$.
15. $3x^2-y^2+18x+15=0$.

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1. 8. 2. 2. 3. 4. 4. 6.

5. $y^2-8x-4y+12=0$. 6. $y^2-20x-2y+41=0$. 7. $y^2+4x+2y+5=0$.

8. $y^2-4x-2y+5=0$. 9. $y^2-8x-4y+12=0$. 10. $(-1, 2)$, $x+2=0$.

11. $(1, 2)$, $x-3=0$. 12. $(2, 1)$, $x=3$. 13. $(0, -1)$, $x+2=0$.

14. $(4, 1)$, y-axis. 15. $(h+a, k)$, $x=h-a$. 16. $(3, 3)$, $y=1$.

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1. $\frac{2}{3}$. 2. $\frac{3}{4}$. 3. $\frac{1}{2}$. 4. $\frac{8}{3}$.

5. $\frac{6}{5}\sqrt{5}$. 6. 2. 7. $(\pm 2, 0)$. 8. $(\pm 3, 0)$.

9. $(\pm\sqrt{3}, 0)$. **10.** $(\pm\sqrt{5}, 0)$. **11.** $(1, 2)$. **12.** $(-2, 1)$.
13. $(0, 2)$. **14.** $(1, -2)$. **15.** $(-3, 1)$, $(1, 1)$.
16. $(0, 1)$, $(-4, 1)$, $2x=5$, $2x+13=0$.
17. $(0, -3)$, $(-5, -3)$, $(5, -3)$. **18.** $(1, 2)$, $y=2$, $x=1$, $\frac{3}{5}$.
19. $(-4, -1)$, $y+1=0$, $x+4=0$, $\frac{1}{2}$.
20. $(-1, -2)$, $y+2=0$, $x+1=0$, $(-4, -2)$, $(2, -2)$.
21. $(-1, 3)$, $\frac{3}{5}$, $(-4, 3)$, $(2, 3)$, $3x+28=0$, $3x=22$.
22. $(-1, \frac{3}{2})$, $\frac{2}{3}$, $(-3, \frac{3}{2})$, $(1, \frac{3}{2})$, $2x+11=0$, $2x=7$.
23. $(3, 2)$, $\frac{2}{3}$, $(1, 2)$, $(5, 2)$, $2x+3=0$, $2x=15$.
24. $\frac{(x-3)^2}{9}+\frac{(y-2)^2}{4}=1$. **25.** $\frac{(x-1)^2}{9}+\frac{(y-1)^2}{5}=1$.
26. $\frac{(x-1)^2}{4}+\frac{(y-1)^2}{2}=1$.

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1. $\frac{5}{4}$. **2.** $\frac{5}{3}$. **3.** 3. **4.** 3.
5. $\frac{4}{3}$. **6.** 1. **7.** $(\pm 3, 0)$. **8.** $(\pm 4, 0)$.
9. $(\pm\frac{3}{2}, 0)$. **10.** $(\pm\frac{5}{2}, 0)$. **11.** $(-1, 2)$. **12.** $(1, -4)$.
13. $(-2, 0)$. **14.** $(2, -1)$. **15.** $2x\pm 3y=0$. **16.** $2x\pm y=0$.
17. $x\pm 3y=0$. **18.** $(-1, 2)$, $\frac{3}{2}$. **19.** $(1, 0)$, $\frac{4}{3}$. **20.** $(-1, 0)$, $\frac{1}{2}\sqrt{6}$.
21. $(1, 0)$, $\frac{3}{2}$. **22.** $3x\pm 5=0$. **23.** $4x\pm 9=0$.
24. $2x-5=0$, $2x+1=0$. **25.** $x=0$, $x+2=0$.
26. $(-1, 2)$, $\frac{4}{3}$, $(3, 2)$, $(-5, 2)$, $4x-5=0$, $4x+13=0$.
27. $(\frac{1}{2}, \frac{2}{3})$, $\frac{4}{3}$, $(\frac{11}{6}, \frac{2}{3})$, $(-\frac{5}{6}, \frac{2}{3})$, $4x-5=0$, $4x+1=0$.
28. $(-2, 1)$, 2, $(2, 1)$, $(-6, 1)$, $x+1=0$, $x+3=0$.
29. $(-2, -1)$, $\frac{5}{4}$, $(3, -1)$, $(-7, -1)$, $5x-6=0$, $5x+26=0$, $3x+4y+10=0$, $3x-4y+2=0$.
30. $\frac{(x-2)^2}{9}-\frac{(y-3)^2}{4}=1$. **31.** $\frac{(x-1)^2}{4}-\frac{(y-2)^2}{12}=1$.
32. $\frac{(x+3)^2}{4}-\frac{(y-2)^2}{12}=1$. **33.** $\frac{(x+1)^2}{4}-\frac{(y-3)^2}{5}=1$.
34. $6(2x+1)^2-(y+1)^2=6$.
35. $\frac{(x-1)^2}{16}-\frac{(y-2)^2}{9}=1$, $3x+4y-11=0$, $3x-4y+5=0$.

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1. $\dfrac{2}{t_1+t_2}$. **9.** $\left(\dfrac{a}{4}, 0\right)$, a.

10. $\left(\dfrac{a}{3}, 0\right)$, $\left(\dfrac{2a}{3}, 0\right)$. **14.** $\left(\dfrac{a^2}{4b}, \dfrac{ca}{2b}\right)$, $\dfrac{c^2}{b}$.

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1. $x-y+a=0$. **2.** $x+4y+4=0$. **3.** $x-2y+3=0$.
4. $(9, 3)$. **5.** $(4, -4)$. **6.** $(5, 2)$.
11. Tangent at vertex.
18. $y=\dfrac{2}{p+q}x+\dfrac{2apq}{p+q}$. **39.** $9x^2-6y+4a=0$.
40. $x+y+a=0$, $x-y+a=0$.
41. $y=\dfrac{x}{t}+at$, $4x+2y+a=0$, $\left(\dfrac{a}{4}, -a\right)$.
44. 1, $-\frac{1}{2}$, $x-y+1=0$, $x+2y+4=0$.

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1. $2x+y-3=0$. **2.** $x-y-3=0$. **3.** $4x+y-22=0$.
4. $(1, -1)$. **5.** $(4, 4)$. **6.** $(5, 2)$.
18. $y+px=2ap+ap^3$. **22.** $y+tx=2at+at^3$.
23. $x+4a=0$, $y^2=16a(x-6a)$, $y^2=2a(x-4a)$.
26. $x-y-3a=0$, $2x-y-12a=0$, $3x+y-33a=0$.
28. $x+y-3a=0$, $4x-8y-9a=0$. **29.** $2a^2$.

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1. $x-2y-2=0$. **2.** $x+y+3=0$. **3.** $x-2y-2=0$.
4. $(0, 2)$. **5.** $(1, 3)$. **6.** $(-1, 2)$.
8. $2x-3y+4=0$, $x-y+1=0$, $x-2y+4=0$. **9.** $x-2y+4=0$.
10. $x+2y+1=0$, $x-6y+9=0$. **12.** $x-2y-3=0$. **13.** $x-2y+2=0$.
14. $x+2y+4=0$. **15.** $(1, 3)$. **16.** $(-2, 1)$.
17. $(2, -1)$. **30.** $y^2+ax=0$. **31.** $x^3=y^2(a-2x)$.

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1. $2x+y-3=0$. 2. $3x+4y-2=0$. 3. $3x-2y-1=0$.

4. $(\frac{3}{2}, 1)$. 5. $(3, -1)$, 6. $(2, 1)$.

8. $\left\{\frac{1}{l^2}(l+2am^2), -\frac{2am}{l}\right\}$. 9. $(2a\tan^2 a+p\sec a, -2a\tan a)$.

10. $y=3a$. 11. $3y+4=0$. 12. $y+3=0$.

13. $-\frac{1}{2}$. 14. $-\frac{1}{4}$. 15. 1.

16. $x-y+a=0$. 17. $4x+4y+1=0$.

18. $2x+6y+3=0$. 22. $\left(\frac{2a}{m^2}-\frac{c}{m}, \frac{2a}{m}\right)$.

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1. $x-y+4=0$. 2. $x-y-3=0$. 3. $2x-y-6=0$.

4. $(-1, -1)$. 5. $(-1, \frac{1}{2})$. 6. $(2, 1)$. 10. $4y=\pm x\pm 5\sqrt{17}$.

21. $\frac{x}{a}(1-t_1t_2)+\frac{y}{b}(t_1+t_2)=1+t_1t_2$; tangents intersect on x-axis.

27. $x-y\pm 3=0$. 28. $2x-y\pm 5=0$. 29. $x+y-4=0$, $x+y+2=0$.

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1. $x-y+1=0$. 2. $2x+3y-1=0$. 3. $x+y-2=0$.

4. $(2, -2)$. 5. $(-1, 2)$. 6. $(4. 3)$.

7. $\frac{2\sqrt{2}ab}{\sqrt{a^2+b^2}}$. 8. $c^2(a^2+m^2b^2)=m^2(a^2-b^2)^2$.

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1. $4x+3y-6=0$. 2. $2x-5y+3=0$. 3. $x-y+1=0$.

4. $(2, 1)$. 5. $(1, -1)$. 6. $(3, 2)$.

7. $x+2y-4=0$, $x=4$, $x-2y+8=0$.

8. $x+y-3=0$, $x-5y+9=0$.

9. $x^2-12xy-6y^2-16x+12y+22=0$. 11. $x+y-5=0$.

12. $3x-2y+9=0$. 13. $2x-2y-1=0$. 14. $(3, 1)$.

15. $(-2, 2)$. 16. $(1, -2)$. 17. $2atx+b(1-t)y-1=0$, $\left(\frac{1}{2a}, \frac{1}{b}\right)$.

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1. $2x+3y-7=0.$ **2.** $3x-4y+7=0.$ **3.** $4x+16y+5=0.$

4. $(1, -1).$ **5.** $(-\frac{1}{2}, \frac{1}{2}).$ **6.** $(0, 1).$

7. $3x+4y=0.$ **8.** $3x-2y=0.$ **9.** $x-3y-1=0.$

10. $-1.$ **11.** $\frac{2}{3}.$ **12.** $-\frac{1}{2}.$ **25.** $y=-\frac{b^2x}{a^2m}.$

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1. $x-2y-2=0.$ **2.** $3x+4y+1=0.$ **3.** $2x-y-1=0.$

4. $(-1, 0).$ **5.** $(-3, -1).$ **6.** $(-1, 2).$

19. $\frac{x}{a}\left(t+\frac{1}{t}\right)-\frac{y}{b}\left(t-\frac{1}{t}\right)=2.$ **20.** $x+y\pm 1=0.$ **21.** $y=\pm x\pm\sqrt{2}.$

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1. $x-y+3=0.$ **2.** $2x-3y-5=0.$ **3.** $4x+6y-25=0.$

4. $(2, 1).$ **5.** $(-2, 1).$ **6.** $(-1, 2).$

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1. $2x-6y-1=0.$ **2.** $x+4y-2=0.$ **3.** $x-3y-2=0.$

5. $x+3y-3=0$; $x+y+1=0$, $3x-y-5=0.$

6. $3x^2+2xy-2y^2-4x-6y-1=0.$

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1. $y+2=0$, $x=1.$

3. $Ax+By=Aa+Bb$, $xy-2x-y=0.$ **5.** $-\frac{1}{t_1t_2}.$

8. $(1, -2)$, $(-\frac{1}{2}, -\frac{1}{2}).$ **10.** $abcd=1.$

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8. $y^2=l(2x+l).$ **15.** $\frac{l}{r}=-e\cos\theta+\cos(\theta-\alpha).$

19. The directrix of the parabola.

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1. Ellipse. **2.** Hyperbola. **3.** Parabola. **4.** Circle.

5. Two intersecting straight lines. **6.** Two parallel straight lines.

7. Rectangular hyperbola.

8. $x^2-4xy+4y^2+6x-28y+41=0$.

10. $(2, -3)$, $x^2-2xy+3y^2=13$. 11. $(1, 2)$, $3x^2+2xy+4y^2+5=0$.

12. 4, 2, $4x-3y=0$, $3x+4y=0$.

13. 6, 4, $2x+y=0$, $x-2y-5=0$.

14. $2\sqrt{2}$, 2, $x+2y-7=0$, $2x-y+1=0$.

15. $2\sqrt{2}$, 2, $4x+3y-5=0$, $3x-4y-10=0$.

16. $2\sqrt{3}$, $2\sqrt{2}$, $x+2y-5=0$, $2x-y=0$.

19. $2x^2-12xy-7y^2-4x-8y-2=0$.

20. $2x^2+xy-y^2-x+2y-1=0$. 21. $x^2-y^2+x+y=0$.

23. $x^2+y^2-6xy+10x+2y-7=0$, $x-y+1=0$, $x+y-3=0$.

24. $2x^2-3xy-2y^2+2x+11y-20=0$.

27. $x^2-6xy+y^2+4x+4y-6=0$. 31. 34.

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1. $y^2-2x-4y+\ =0$. 2. $y^2+8x-2y-39=0$.

3. $x^2-2xy+y^2+12y+6=0$. 4. $(1, 0)$, $(\frac{3}{2}, 0)$.

5. $(3, 1)$, $(5, 1)$. 6. $(-1, 1)$, $(-\frac{5}{4}, 1)$. 9. $(a, 0)$.

15. $\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$, $\pm\frac{1}{a}$. 19. $2p$. 21. $a^2(t_1-t_2)(1+t_1t_2)$.

29. $4a\cos(\theta+\phi)\sec\theta\operatorname{cosec}^2\phi$. 36. $p+aq^2=0$. 37. $AC=aB^2$.

38. $x_1\tan^2\theta-y_1\tan\theta+a=0$. 39. $4x+2y+a=0$, $4x-2y+a=0$.

40. $x+y+a=0$, $x-y+a=0$. 41. Tangent at vertex.

51. $yy_1=2a(x+x_1)$, $y-y_1=-\frac{y_1}{2a}(x-x_1)$.

60. $4x+4y-3=0$, $2x-y-3=0$.

62. $4x=a$. 76. $(t_1+t_2)y=2x+2at_1t_2$.

79. $5x^2+9y^2+14x-55=0$. 80. $x+2=0$, $x=6$, $\frac{1}{2}$.

88. $3x^2+4y^2+6x-9=0$, $(-1, 0)$, $\frac{1}{2}$. 95. $\frac{4}{5}$.

104. $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$, $\frac{a^2\cos\alpha}{p}$. $\frac{b^2\sin\alpha}{p}$.

109. $\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$,

$$a\frac{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}},\quad b\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}.$$

119. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2+y^2}{a^2+b^2}$. **158.** $a(t_1^2+t_1t_2+t_2^2+2)$, $-at_1t^2(t_1+t_2)$.

173. $4r^2-4\beta r\cos(\theta-\alpha)+\beta^2-a^2=0$, $\frac{a}{2}$.

188. Ellipse. **189.** Circle. **190.** Point. **191.** Hyperbola.

192. Two intersecting straight lines. **193.** Rectangular hyperbola.

194. Parabola. **195.** Two parallel straight lines.

196. $(1, 2)$, $x-2y+3=0$, $2x+y-4=0$, $4\sqrt{5}$, $2\sqrt{5}$.

197. $(1, 1)$, $x-3y+2=0$, $3x+y-4=0$, $2\sqrt{10}$, $\sqrt{10}$, $x-y=0$, $x+7y-8=0$.

198. $(1, 0)$, $x-3y-1=0$, $3x+y-3=0$, $2\sqrt{10}$, $2x-y-2=0$, $x+2y-1=0$.

199. $(4, 1)$, $2x+y-4=0$.

200. $(pt_1+q)(pt_2+q)+(at_1+b)(at_2+b)$
$$+\{(at_1+b)(pt_2+q)+(at_2+b)(pt_1+q)\}\cos\omega=0.$$

201. $3x^2-8xy-3y^2+22x+54y-143=0$.

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